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1. *Introduction.*

IN my recent paper † I brought forward a hypothesis according to which the elementary particles (*e. g.*, protons, electrons, positrons, etc.) were assumed to be certain axially symmetrical systems of Maxwell electromagnetic waves *in vacuo*, which were called "electromagnetic whirls," or simply "Whirls."

* Communicated by Sir W. H. Bragg, O.M., K.B.E., M.A., Pres.R.S.

† Phil. Mag., Sept. 1935, pp. 417-468; Oct. 1935, pp. 641-706.

It was shown in that paper, on the basis of Maxwell Theory, that the Whirls possess fundamental features ascribed to the elementary particles, classical as well as wave mechanical and relativistic.

The Whirls which were interpreted as the proton and electron differ only in dimensions, which are inversely proportional to their masses. The particles of the same dimensions and masses as the proton and the electron, but with the mutually interchanged positions of the electric and magnetic lines of force, are called the "magneto-electron" and "magneto-proton" resp. The proton and the electron are called "electropolarized" particles, and generally Whirls with a similar position of the electric and magnetic lines of force are called "electropolarized Whirls." Correspondingly, the magneto-proton and the magneto-electron are called "magnetopolarized particles" and the corresponding Whirls "magnetopolarized Whirls."

The particles, and generally the Whirls, with similar positions of electric and magnetic lines (*e. g.*, the electron and the proton, or the magneto-electron and the magneto-proton) are called "copolarized"; those with mutually interchanged positions of electric and magnetic lines (*e. g.*, the electron and the magneto-electron or the magneto-proton) are called "contrapolarized."

The copolarized particles, on the average, and when the distance between their centres is not too small, exercise the usual electrostatic forces (Coulomb forces), and their axes tend to be parallel.

The axes of contrapolarized particles, on the contrary, tend to be mutually perpendicular. When the masses of two contrapolarized particles differ greatly (in other words, when they are of opposite signs) they attract each other according to the inverse square law, but with a much smaller force than the copolarized particles of opposite signs. When two contrapolarized particles have equal masses, *i. e.*, are of the same sign, they attract each other in accordance with the inverse *cube* law, and their interaction can be neglected unless the distance between their centres is very small, in which case their mutual attraction can be very strong.

The magneto-electron was identified with the positron, while the neutron was assumed to be a hydrogen-like

combination of the magneto-electron (positron) and the magneto-proton*.

It was suggested that the magnetopolarized particles (*i. e.*, the positron and the magneto-proton) may "act as a cement" in the structure of nuclei †.

The purpose of the present work is to develop this suggestion a little further, and to show how the structures of certain simplest nuclei can be visualized on the basis of the above-mentioned theory of elementary particles.

The present paper must be read in conjunction with the above-cited previous paper. To facilitate the references we shall use here for the formulæ taken from the latter paper the same reference numbers as there, with the sole difference that these reference numbers will be put in [] in addition to () in which they were enclosed. The reference numbers enclosed only in () will relate to formulæ in the present paper. We shall also use the same notation in the present paper as in the above-cited one, unless otherwise stated.

2. Interaction between two Contrapolarized Whirls of Equal Size.

The question of interaction between contra-polarized Whirls of equal size was already discussed in the above-cited paper, and as a result of this discussion the *inverse cube* law of attraction was derived (formula [(19.9b)]) ‡. This result is, however, insufficient, since the constant N_4 has been left undetermined, as the knowledge of its numerical value was then not essential. Here we shall have to complete this work.

We shall use the same methods as we generally used in sections 18 and 19 of the above-cited paper.

Since both particles are identical, we can take, at least approximately, that

$$q'_{01} = q'_{02} \S. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.1)$$

* This hypothesis will be slightly modified (see section 7, p. 573).

† *Loc. cit.* p. 701.

‡ There is a misprint in this formula, which is obvious from the preceding text; it should be D^3 and *not* D^2 in the denominator.

§ *Loc cit.*, the text between [(16.1)] and [(16.2)].

Further, by [(18.6 *a*)] *

$$\theta_1 = 0, \quad \theta_2 = \frac{\pi}{2}. \quad . \quad . \quad . \quad . \quad (2.2)$$

Following the Minimum Disturbance Principle (see the text between [(18.7 *b*)] and [(18.8)]), and from the symmetry consideration,

$$\varphi_2 = \frac{\pi}{2}, \quad . \quad . \quad . \quad . \quad . \quad (2.3)$$

which corresponds to the case where the axes of both Whirls intersect.

By (2.1)–(2.3) we obtain the following expressions from [(16.4)], [(16.5)], and [(16.5 *a*)], on assuming that the speed of translatory motion of the Whirl is small in comparison with the speed of light, and putting, therefore, $\beta=1$ (whatever be the suffix) †,

$$\left. \begin{aligned} (X_{01}^2 + Y_{01}^2)(X_{02}^2 + Y_{02}^2)_{000} &= \frac{1}{16} D^4, \\ [Z_{01}]_{000} &= [Z_{02}]_{000} = 0, \\ (\cos \theta_2)_{000} &= 1, \end{aligned} \right\} . \quad (2.4)$$

where *D* is the distance between the centres of the Whirls. The triple suffix ₀₀₀ is to show that in the expression to which they relate *X*, *Y*, and *Z* are equalled to zero ‡.

From (2.4), [(17.10 *b*)], and [(17.5 *a*)],

$${}^4u_{000} = u_{000} = 8D^{-3}. \quad . \quad . \quad . \quad . \quad (2.5)$$

By [(18.11)], (2.2)§, and (2.3)

$$\left. \begin{aligned} L_{1fg} &= q'_{01} \left(\sqrt{2} \cos \left(\varphi_1 \pm \frac{\pi}{4} \right) + (-)^g v_2 \right), \\ L_{2fg} &= q'_{01} \left(\sqrt{2} \sin \left(\varphi_1 \pm \frac{\pi}{4} \right) + \bar{f} \right), \\ L_{3fg} &= \bar{f} q'_{01}, \\ L_{4fg} &= \frac{q'_{01}}{2} \left(\sqrt{2} \sin \left(\varphi_1 \pm \frac{\pi}{4} \right) - \bar{f} \right). \end{aligned} \right\} . \quad (2.6)$$

* For the definition of θ and φ see the text between [(16.3)] and [(16.3 *a*)].

† See formulæ [(16.3 *a*)], [(16.7)], [(16.7 *a*)], and [(18.1 *d*)].

‡ For the definition of X_{01} , X_{02} etc., see the text between formulæ [(16.3)] and [(16.3 *a*)]. The meaning of the triple suffixes is fully explained in the text between formulæ [(17.10 *d*)] and [(17.11)].

§ For the definition of $\bar{f}$ and $\bar{g}$ see [(17.7 *c*)], and for the general definition of L_{1fg} etc. see [(17.10 *c*)].

We shall limit our consideration to the case where the speed of the Whirls is sufficiently small in comparison with the speed of light to enable us to put $v_2=0$ (see formula [(16.6b)]* and the paragraph following formula [(18.1 d)]).

This will be sufficient for our purpose, since the force between the contrapolarized particles of equal size is appreciable only when they are situated very near to each other and the linear velocities of their relative motion are comparatively small (see below). Thus on putting $v_2=0$ we shall have from (2.6), taking into account that $\bar{f}=\pm 1$ (see [(17.7 c)]),

$$\frac{|L_4|}{|L_1 L_2 L_3|} = \frac{1}{2\sqrt{2}q_{01}'^2 \left| \cos\left(\varphi_1 \pm \frac{\pi}{4}\right) \right| \left| \sin\left(\varphi_1 \pm \frac{\pi}{4}\right) \mp \sin \frac{\pi}{4} \right| \left| \sin\left(\varphi_1 \pm \frac{\pi}{4}\right) \pm \sin \frac{\pi}{4} \right|}, \quad (2.7)$$

where $| |$ are to show that only the modulus (which in the present case means the absolute value irrespective of sign) of the quantity in $| |$ must be taken.

By (2.5), (2.7), and [(18.11 b)]

$$\left. \begin{aligned} \frac{|L_{4fg}|}{q_{01}=q_{02} > > 1} \frac{W_-}{-} &= -K_1 K_2 \frac{1}{q_{01}' D^3} \bar{N}_4^2, \\ f &= 3, 4, \\ g &= 3, 4, \end{aligned} \right\} \quad (2.8)$$

where K_1 and $K_2^\dagger$ are constants, and

$$N_4^2 = 2\sqrt{2} \left| \frac{\sin \Phi_{fg000}}{\cos\left(\varphi_1 \pm \frac{\pi}{4}\right)} \frac{\left| \sin\left(\varphi_1 \pm \frac{\pi}{4}\right) \mp \sin \frac{\pi}{4} \right|}{\left| \sin\left(\varphi_1 \pm \frac{\pi}{4}\right) \pm \sin \frac{\pi}{4} \right|} \right|. \quad (2.9)$$

Here the bars on top mean that the average value must be taken, and thick straight brackets $| |$ mean that the quantity inside these brackets must be positive (in contradistinction of thin straight brackets $| |$, which mean that only the modulus of the quantity is taken, but does not exclude the possibility of the quantity being negative) ‡ .

* There is an obvious misprint in that formula: it should be v_1 and not v .

† These constants are defined by formulæ [(17.3 a)].

‡ Compare with the paragraph after formula [(18.7)]

By (2.8), [(17.11 d)] and [(19.1)], after simple transformation,

$$F_{D-} = \frac{K_1 K_2}{q_{01}' D^3} \left(\bar{N}_4^2 - \left(\frac{\bar{N}_4^2}{|L_4|} \right) \frac{3}{D} \right). \quad (2.10)$$

On comparing (2.10) with [(19.9 b)], and recalling that in the latter formula the quantities of the relative order $(q_0 D)^{-1}$ were neglected, we see that $\bar{N}_4^2$ in (2.9)–(2.10) denotes the same thing as in [(19.9 b)]*.

It follows from (2.8) that the *mutual*† energy W will remain finite at every moment only if $\frac{\bar{N}_4^2}{|L_4|}$ will remain finite. Using exactly the same reasoning as we used when deriving the expression [(19.9 f)] for $\bar{N}_1^2$ ‡ we see from (2.9) and the fourth formula (2.6) that one of the simplest methods of satisfying the condition of finiteness of $\frac{\bar{N}_4^2}{|L_4|}$ is to assume that

$$\sin \Phi_{f,000} = \sin \left| a \cos \left(\varphi_1 \pm \frac{\pi}{4} \right) \right| \cdot \left| \left(\sqrt{2} \sin \left(\varphi_1 \pm \frac{\pi}{4} \right) + \bar{f} \right) \right|. \quad (2.10 a)$$

Continuing with the same reasoning, we shall find by [(19.9 g)], [(19.9 h)], and (2.9), that $a \simeq 1.65$ and

$$\frac{q_{01}}{2} \left(\frac{\bar{N}_4^2}{|L_4|} \right) = \frac{4\bar{N}_1^2}{\pi} = \frac{4 \times 2.52}{\pi} \simeq 3.2. \quad (2.10 b)$$

To calculate approximately $\bar{N}_4^2$ we shall assume that

$$\bar{N}_4^2 = |L_4| \left(\frac{\bar{N}_4^2}{|L_4|} \right). \quad (2.11)$$

On finding from the fourth formula (2.6) and from the

* The misprint, mentioned in the footnote at the beginning of the present section, must be taken into account. Also in [(19.9 b)] q_0 is taken in the denominator instead of q_0' , but here for all practical purposes we can take $q_0' = q_0$.

† See section 15, *loc. cit.*

‡ Pp. 686–688, *loc. cit.*

second formula [(19.10 e)] that $|\overline{L}_4| = \frac{q'_{01}}{2}$, we shall easily obtain from (2.10 b) and (2.11)

$$\overline{N}_4^2 = \frac{4\overline{N}_1^2}{\pi} \approx 3 \cdot 2^*. \quad (2.11 a)$$

By (2.10), (2.10 b), and (2.11 a)

$$\begin{aligned} F_{D-} &= \frac{4K_1 K_2 \overline{N}_1^2}{\pi q'_{01} D^3} \left(1 - \frac{6}{q'_{01} D} \right) \\ &= \frac{-4}{\pi q'_{01} D} F_{D+} \left(1 - \frac{6}{q'_{01} D} \right), \end{aligned} \quad (2.12)$$

by [(19.3b)].

Hence, applying this result to elementary particles, we obtain by [(19.11)] and [(20.3)]

$$\left. \begin{aligned} F_{D-}^{(++)} &= \frac{4}{\pi q_{0+} D} \frac{e^2}{D^2} \left(1 - \frac{6}{q_{0+} D} \right) \\ &= \frac{6 \times 10^{-33}}{D^3} \left(1 - \frac{1 \cdot 24 \times 10^{-13}}{D} \right), \\ F_{D-}^{(--)} &= \frac{4}{\pi q_{0-} D} \frac{e^2}{D^2} \left(1 - \frac{6}{q_{0-} D} \right) \\ &= \frac{1 \cdot 108 \times 10^{-29}}{D^3} \left(1 - \frac{2 \cdot 24 \times 10^{-10}}{D} \right), \end{aligned} \right\} \quad (2.12 a)$$

where, in conformity with the previous notation, $F_{D-}^{(++)}$ and $F_{D-}^{(--)}$ denote the forces of attraction between the proton and magneto-proton, and between the electron and magneto-electron (positron) resp.

As has been already mentioned (see formula (2.2)), the axes of the contrapolarized particles tend to be mutually perpendicular.

Therefore we cannot expect in the case of two contrapolarized particles of equal size continuous planetary motion as in the case of copolarized particles (the axes of which tend to be parallel[†]), for when the axes are

* In calculating $\overline{N}_4^2$ more rigorously, i. e., by direct averaging the expression for N_4^2 in the same manner as we did when deriving the expression for $\overline{N}_1^2$, (see formula [(19.9 h)]), I have obtained practically the same numerical result.

† See formula [(18.5)].

mutually perpendicular the orbits of this motion cannot be similarly situated in both particles in relation to their respective axes. On the other hand, there is no reason why these orbits with respect to one of these particles should be situated differently from the other.

Consequently, in the case of two contrapolarized particles of the same sign, we must expect an oscillation about their positions of equilibrium, but not a planetary motion.

By (2.12a) the positions of equilibrium ($F_{D-}=0$) will correspond to the distances

$$\left. \begin{aligned} D_{0+} &= \frac{6}{q_{0+}} = 1.24 \times 10^{-13} \text{ cm.}, \\ D_{0-} &= \frac{6}{q_{0-}} = 2.29 \times 10^{-10} \text{ cm.} \end{aligned} \right\} \dots (2.12b)$$

By the same formula (2.12a) the maximum force of attraction

$$\frac{\partial F_{D-}}{\partial D} = 0, \quad \frac{\partial^2 F_{D-}}{\partial D^2} < 0,$$

will correspond to the distance

$$\left. \begin{aligned} D_m &= \frac{4}{3} D_0 = \frac{8}{q_{0+}} = 1.65 \times 10^{-13} \text{ cm.}, \\ D_m &= \frac{4}{3} D_0 = \frac{8}{q_{0-}} = 3.04 \times 10^{-10} \text{ cm.} \end{aligned} \right\} (2.12c)$$

By (2.12a) and (2.12c) the value of this maximum force

$$\left. \begin{aligned} F_{Dm-} &= 3.33 \times 10^5 \text{ dyn.}, \\ F_{Dm-} &= 9.76 \times 10^{-2} \text{ dyn.} \end{aligned} \right\} \dots (2.12d)$$

By (2.12a) and (2.12b) the gradient of force at the equilibrium position can be expressed as follows :

$$\begin{aligned} \left(\frac{dF_{D-}}{d\delta} \right)_{\delta=\delta_0} &= \frac{4e^2 q_0^2}{\pi \delta_0^4} \left(-3 + \frac{24}{\delta_0} \right) \\ &= \frac{e^2}{324\pi \hbar c} q_0^2 \hbar c = 7.284 \times 10^{-6} q_0^2 \hbar c, \end{aligned} \quad (2.12e)$$

by [(20.1)]-[(20.3)].

Here $\delta = q_0 D$, $\delta_0 = q_0 D_0 = 6$.

The form given in the last member will be convenient in the future. The suffix at the q_0 must be $+$ or $-$, depending on the sign of particles in question.

The energy E , required to separate two particles in equilibrium, will be found from (2.12 *a*) by integration with respect to D from D_0 (equilibrium) to infinity.

Hence by (2.12 *a*) and [(20.1)]-[(20.3)]

$$\begin{aligned} E_{s-}^{(+)} &= \frac{4}{\pi q_{0+}} \frac{e^2}{D_{0+}^2} \left(\frac{1}{2} - \frac{2}{q_{0+} D_{0+}} \right) \\ &= \frac{e^2 q_{0+}}{54\pi} = 6.5 \times 10^{-8} \text{ erg} = 40650 \text{ el. volt} \end{aligned}$$

(corresponding to the wave-length

$$\lambda_{s-}^{(+)} = \frac{2\pi c \hbar}{E_{s-}^{(+)}} = 2.97 \times 10^{-9} \text{ cm.} = 0.297 \text{ \AA.},$$

$$E_{s-}^{(-)} = \frac{e^2 q_{0-}}{54\pi} = 3.512 \times 10^{-11} \text{ erg} = 22 \text{ el. volt}$$

(corresponding to the wave-length

$$\lambda_{s-}^{(-)} = 5.5 \times 10^{-6} \text{ cm.} = 55 \text{ \AA.} \quad (2.12 f)$$

These figures suggest that a combination of two positive contrapolarized particles (proton, magneto-proton) must be much more stable than a combination of two negative ones (electrons, magneto-electron, positron).

Combinations ("pairs") of contrapolarized particles, which were considered above, having the axis of symmetry of one of the particles lying in the equatorial ("central") plane of the other while the axis of symmetry of the latter is parallel to, but does not lie in, the central plane of the former (fig. 1 (*a*)), are not likely to occur unless there is some factor owing to which the axis of one of the particles will be the axis of symmetry of the whole system in preference to the other.

In the absence of such a factor a combination in which the axes intersect like the axes of two bevels is more probable (fig. 1 (*b*)). In the latter case, however, at a

glance the system does not appear to be as stable as in the case corresponding to fig. 1 (a).

We shall limit ourselves to this remark, and shall not consider the system corresponding to fig. 1 (b) any further here.

3. *Interaction between Copolarized and Co-axial Identical Whirls.*

In the above-cited paper the problem of interaction between two Whirls was limited to the case where the axis of at least one of the Whirls has a precessional rotation about an axis perpendicular to the line joining the centres of the Whirls.

Fig. 1.

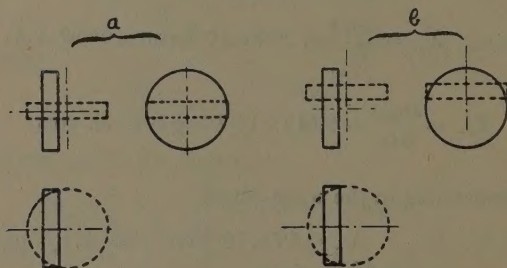


Diagram showing two relative positions of contrapolarized whirls of equal size.

- (a) The axis of the one is situated *in* the central plane of the other.
- (b) The axis of the one is parallel to the central plane of the other, but lies outside it.

The electropolarized and magnetopolarized particles are represented in the form of disks drawn in solid and dotted lines resp.

This obviously does not include the case where both Whirls remain co-axial.

We shall consider this case separately in the present section.

As the coordinate system common for both Whirls we shall use a co-axial cylindrical coordinate system.

In accordance with the general rule given at the beginning of section 16 of the above-cited paper* we shall place the origin of this common coordinate

* *Loc. cit.* p. 648.

system at the mid-point between their centres. Then we shall have

$$\left. \begin{aligned} \rho_1 = \rho_2 = \rho, \\ \xi_1 = \xi - \frac{q_0 D}{2}, \quad \xi_2 = \xi + \frac{q_0 D}{2}, \end{aligned} \right\} \quad (3.1)$$

where $\rho = q_0 r$ and $\xi = \beta q_0 |x|$ are the radial and the axial coordinates in the coordinate system common for both Whirls, expressed in special abstract units *, while r and x are the same coordinates expressed in ordinary units, ξ_1 and ξ_2 denote the ξ of Whirls 1 and 2 resp. in relation to their appropriate cylindrical coordinate systems (with the origins at their respective centres) as defined by formulæ [(4.12)] and [(5.4)].

We shall now write here the first formula [(17.2)] † in cylindrical (x, r, θ) instead of rectangular coordinates XYZ. We shall have

$$W_+ = \frac{1}{4\pi} \beta_1 \beta_2 A_1 A_2 q_0^4 \int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{2\pi} (\vec{L}_1 \vec{L}_2' + \vec{M}_1 \vec{M}_2') r dx dr d\theta, \quad (3.2)$$

where, by analogy with [(17.2a)], and in accordance with [(8.4)], $\vec{L}_1$, $\vec{L}_2'$, $\vec{M}_1$, and $\vec{M}_2'$ are the vectors having the following components :

$$\left. \begin{aligned} L_{1x}' &= R(-L_{1x} e^{i\phi_1}), & M_{1x}' &= R(-iM_{1x} e^{i\phi_1}), \\ L_{1r}' &= R(iL_{1r} e^{i\phi_1}), & M_{1r}' &= R(-M_{1r} e^{i\phi_1}), \\ L_{1\theta}' &= R(L_{1\theta} e^{i\phi_1}), & M_{1\theta}' &= R(iM_{1\theta} e^{i\phi_1}). \end{aligned} \right\} \quad (3.2a)$$

We shall limit ourselves to the case where

$$\beta_1 \cong \beta_2 = 1, \quad A_1 = A_2 = A, \quad n_1 = n_2 = 1. \quad (3.3)$$

Then, using the notation $\rho = q_0 r$ and $\xi = \beta q_0 |x|$, and taking into account that the subintegral expression is an *even* ‡ function of x , we shall have from (3.2)

$$W_+ = \frac{A^2 q_0}{2\pi} \int_0^{\infty} \int_0^{\infty} \int_0^{2\pi} (\vec{L}_1 \vec{L}_2' + \vec{M}_1 \vec{M}_2') \rho d\xi d\rho d\theta. \quad (3.4)$$

* See formulæ [(4.12)] and [(5.4)].

† The formulæ [(17.2)] were misprinted, namely, the signs of $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty}$ before the binomials in () and $dX dY dZ$ after these binomials were missing. This is quite obvious at a glance from a comparison of these formulæ with [(17.1)].

‡ This will be seen more clearly in the course of further discussion.

Since we shall deal in this section only with the case $q_{01}=q_{02}=q_0$, we need not write this suffix under W_+ .

By (3.4) and (3.2 a)

$$W_+ = -\frac{A^2 q_0}{2\pi} \int_0^\infty \int_0^\infty \int_0^{2\pi} [(L_{1x}L_{2x} + L_{1\theta}L_{2\theta} + M_{1r}M_{2r}) \cos \phi'_1 \cos \phi'_2 \\ + (L_{1x}L_{2r} + M_{1x}M_{2x} + M_{1\theta}M_{2\theta}) \sin \phi'_1 \sin \phi'_2] \rho \, d\xi \, d\rho \, d\theta. * \\ \dots (3.4 a)$$

By [(16.6 a)] and (3.3)

$$\phi'_1 = q_0 ct + \theta_1 + \psi_1, \quad \phi'_2 = q_0 ct + \theta_2 + \psi_2. \quad (3.5)$$

We can take a common initial radius for θ_1 and θ_2 , and thus put

$$\theta_1 = \theta_2 = \theta. \quad (3.5 a)$$

Then ψ_1 and ψ_2 can be considered as the initial phases (which, generally speaking, can fluctuate). We can thus write (3.5) in the form

$$\phi'_1 = q_0 ct + \theta + \psi_1, \quad \phi'_2 = q_0 ct + \theta + \psi_2. \quad (3.5 b)$$

We shall limit ourselves to the case where ψ_1 and ψ_2 are either constant or oscillate harmonically with a small amplitude about their average values.

Then on introducing (3.5 b) into (3.4 a), performing the integration with respect to θ , and taking root mean square value with respect to time t , we shall obtain for the average of W (potential energy)

$$\bar{W}_+ = -\frac{A^2 q_0}{4} \overline{\cos \psi} \int_0^\infty \int_0^\infty (L_{1x}L_{2x} + L_{1r}L_{2r} + L_{1\theta}L_{2\theta} \\ + M_{1x}M_{2x} + M_{1r}M_{2r} + M_{1\theta}M_{2\theta}) \rho \, d\xi \, d\rho, \quad (3.6)$$

where $\psi = \psi_1 - \psi_2 \quad \dots \dots (3.6 a)$

and $\overline{\cos \psi}$ is the average value of $\cos \psi$.

In conformity with [(18.4 d)] the stable state will correspond to $\bar{W}_+ < 0$ and the maximum stability with respect to ψ will be obtained at a maximum absolute value of $\overline{\cos \psi}^\dagger$. We shall assume therefore that,

* It should be noted that, generally speaking, M_{1r} etc. are not always real quantities (see formulae [(8.10)]-[(8.12)]). We shall see, however, that this will not affect the value of the integral in the cases which will interest us.

† See the text preceding formula [(19.9 d)].

in the absence of an external influence* (such as the presence of a third Whirl),

$$|\overline{\cos \psi}| \cong 1, \quad . \quad . \quad . \quad . \quad . \quad (3.6b)$$

and write (3.6) in the form

$$\overline{W}_+ = -A^2 q_0 |\mathbf{Q}_{(12)}|, \quad . \quad . \quad . \quad . \quad (3.7)$$

where

$$\mathbf{Q}_{(12)} = \frac{1}{4} \int_0^\infty \int_0^\infty (\vec{L}_1 \vec{L}_2 + \vec{M}_1 \vec{M}_2) \rho d\xi d\rho, \quad . \quad (3.7a)$$

where $\vec{L}_1$, $\vec{M}_1$, etc. are vectors (L_{1x} , L_{1r} , $L_{1\theta}$) and (M_{1x} , M_{1r} , $M_{1\theta}$) etc. resp.

By comparison with section 9 of the above-cited paper $\mathbf{Q}_{(12)}$ plays the same part in relation to the *mutual* energy $\overline{W}_+$ of two co-axial and identical Whirls as $\mathbf{Q}$ plays in relation to the *internal* energy of a single Whirl (see formulæ [(9.1)] and [(9.2)] in the case of $\beta \cong 1$).

To determine $\mathbf{Q}_{(12)}$ we shall use a method similar to that which was used for determining $\mathbf{Q}$, namely, we shall put †

$$\mathbf{Q}_{(12)} = \mathbf{Q}_{(12)1} + \mathbf{Q}_{(12)2}, \quad . \quad . \quad . \quad . \quad (3.8)$$

where

$$\left. \begin{aligned} \mathbf{Q}_{(12)1} &= \frac{1}{4} \int_0^\infty \int_0^{\rho_1} (\vec{L}_1 \vec{L}_2 + \vec{M}_1 \vec{M}_2) \rho d\xi d\rho, \\ \mathbf{Q}_{(12)2} &= \frac{1}{4} \int_0^\infty \int_{\rho_1}^\infty (\vec{L}_1 \vec{L}_2 + \vec{M}_1 \vec{M}_2) \rho d\xi d\rho, \end{aligned} \right\} \quad (3.8a)$$

where ρ_1 corresponds to the radius of a cylinder beyond which the approximate formulæ [(8.10)]–[(8.12)] can be used for the calculation of $\vec{L}$ and $\vec{M}$.

To obtain the final expressions for $\mathbf{Q}_{(12)1}$ and $\mathbf{Q}_{(12)2}$ we can introduce into the first and the second formulæ (3.8a) the expressions for L_1 , L_2 , etc. drawn from [(8.5)] and [(8.10)]–[(8.12)] resp.

For the reasons stated in the above-cited paper ‡ $\mathbf{Q}_{(12)1}$ must be much smaller than $\mathbf{Q}_{(12)2}$; therefore we can concentrate on the latter.

* See formula (3.12a) and section 8.

† Compare with formulæ [(9.5)] and [(9.6)].

‡ Pp. 441 and 448, *loc. cit.*

We shall limit ourselves to the case where $\chi_1 = \chi_2 \ll 1$ *. Then from the second formulæ (3.8a) and [(8.10)]-[(18.12)], taking into account (3.1) and changing the variables ξ and ρ for

$$\xi' = \frac{2\xi}{q_0 D} = \frac{2\beta |x|}{D} \quad \text{and} \quad \rho' = \frac{2\rho}{q_0 D} = \frac{2r}{D}, \quad (3.8b)$$

and putting

$$\rho_1' = \frac{2\rho_1}{q_0 D},$$

we shall find the following expression:

$$\begin{aligned} Q_{(12)2} &\simeq \frac{2}{\pi} \left(e^{-\chi_1 q_0 D} \int_0^1 + \int_1^\infty e^{-\chi_1 q_0 D \xi'} \right) \\ &\quad \times \int_{\rho_1'}^\infty \frac{d\xi' d\rho'}{\sqrt{[\rho'^2 + (\xi' - 1)^2][\rho'^2 + (\xi' + 1)^2]}} \\ &= \frac{2}{\pi} e^{-\chi_1 q_0 D} \left(\int_{-1}^0 + \int_0^1 e^{-\chi_1 q_0 D \xi'} \right) \\ &\quad \times \int_{\rho_1'}^\infty \frac{d\xi' d\rho'}{\sqrt{(\rho'^2 + \xi'^2)[\rho'^2 + (\xi' + 2)^2]}}, \quad (3.8c) \end{aligned}$$

by substituting ξ' for $\xi' - 1$.

The integral with respect to ρ' can be easily transformed into a standard form of elliptic *integrals of the first kind* †. We shall have

$$\begin{aligned} &\int_{\rho_1'}^\infty \frac{d\rho'}{\sqrt{(\rho'^2 + \xi'^2)[\rho'^2 + (\xi' + 2)^2]}} \\ &= \frac{1}{\xi' + 2} \left| \operatorname{tn}^{-1} \left[\frac{\rho'}{\xi'}, \frac{2\sqrt{\xi' + 1}}{\xi' + 2} \right] \right|_{\rho'=\rho_1'}^{\rho'=\infty}. \quad (3.8d) \end{aligned}$$

The integration can be then performed either by using the Tables of Elliptic Functions or by expanding the tn^{-1} into a series ‡.

When, however, $\chi_1 q_0 D \ll 1$, as, indeed, will be the case in our applications in the present paper, it will be more convenient to use a simpler method of estimating the

* See formulæ [(18.1 c)].

† Compare [(19.4b)] and [(19.4 c)]. See also Table of Integrals etc. by H. B. Dwight (Macmillan, New York, 1934), specially arts. 750, 752, and 780.3.

‡ See H. B. Dwight, *loc. cit.*, arts. 773.1-773.3, 775, and 776.

integrals in (3.8 c), which will enable us to use for our calculations of $Q_{(12)2}$ some of the previously obtained results.

We have

$$\begin{aligned} & \left(\int_{-1}^0 + \int_0^\infty e^{-\chi_1 q_0 D \xi'} \right) \int_{\rho_1'}^\infty \frac{d\xi' d\rho'}{\rho'^2 + (\xi' + 2)^2} \\ & < \left(\int_{-1}^0 + \int_0^\infty e^{-\chi_1 q_0 D \xi'} \right) \int_{\rho_1'}^\infty \frac{d\xi' d\rho'}{\sqrt{(\rho'^2 + \xi'^2)[\rho'^2 + (\xi' + 2)^2]}} \\ & < \left(\int_{-1}^0 + \int_0^\infty e^{-\chi_1 q_0 D \xi'} \right) \int_{\rho_1'}^\infty \frac{d\xi' d\rho'}{\rho'^2 + \xi'^2} \dots \dots \dots (3.9) \end{aligned}$$

Hence, on integrating with respect to ρ and substituting in the first member ξ' for $\xi' + 2$ *,

$$\begin{aligned} & \left(\int_1^2 + e^{2\chi_1 q_0 D} \int_2^\infty e^{-\chi_1 q_0 D \xi'} \right) \left(\frac{\pi}{2} - \tan^{-1} \frac{\rho_1'}{\xi'} \right) \frac{d\xi'}{\xi'} \\ & < \left(\int_{-1}^0 + \int_0^\infty e^{-\chi_1 q_0 D \xi'} \right) \int_{\rho_1'}^\infty \frac{d\xi' d\rho'}{\sqrt{(\rho'^2 + \xi'^2)[\rho'^2 + (\xi' + 2)^2]}} \\ & < \left(\int_{-1}^0 + \int_0^\infty e^{-\chi_1 q_0 D \xi'} \right) \left(\frac{\pi}{2} - \tan^{-1} \frac{\rho_1'}{\xi'} \right) \frac{d\xi'}{\xi'} \dots \dots \dots (3.9a) \end{aligned}$$

Since $\chi_1 q_0 D \ll 1$ and also $\chi_1 q_0 \ll 1$, we can put

$$\left. \begin{aligned} e^{\chi_1 q_0 D} &\cong e^{-\chi_1 q_0 D} \cong 1 \\ e^{-\chi_1 q_0 \xi'} &\cong 1, \end{aligned} \right\} \dots \dots \dots (3.9b)$$

and

when $-1 \leq \xi' \leq 2$.

The difference between the two limits in (3.9 a) will be approximately equal

$$\begin{aligned} & \left(\int_{-1}^0 + \int_0^1 \right) \left(\frac{\pi}{2} - \tan^{-1} \frac{\rho_1'}{\xi'} \right) \frac{d\xi'}{\xi'} \\ & = \int_{-1}^1 \frac{\tan^{-1} \frac{\xi'}{\rho_1'}}{\xi'} d\xi' = 2 \sum_{m=0}^{\infty} (-)^m \frac{\rho_1'^{1-2m-1}}{m^2} \dots \dots \dots (3.9c) \end{aligned}$$

In our calculation $\rho' = \frac{\rho}{q_0 D}$ will be of the order of unity, while the value of the integral in (3.8 c) will be about 100 or more†. We can take, therefore, for our

* See *loc. cit.*, formula [(9.7)] and the footnote on p. 698.

† See section 6.

integral an average between these two limits, and hence, by (3.8 c) and (3.9), we can put

$$\begin{aligned} Q_{(12)2} &= \frac{2}{\pi} \left(\int_{-1}^0 + \int_0^{\infty} e^{-\chi_1 q_0 D \xi'} \right) \int_{\rho_1'}^{\infty} \frac{d\xi' d\rho'}{\rho'^2 + (\xi' + 1)^2} \\ &\approx \frac{2}{\pi} \int_{-1}^{\infty} \int_{\rho_1'}^{\infty} \frac{e^{-\chi_1 q_0 D \xi'} d\xi' d\rho'}{\rho'^2 + (\xi' + 1)^2} \approx \frac{2}{\pi} \int_0^{\infty} \int_{\rho_1'}^{\infty} \frac{e^{-\chi_1 q_0 D \xi'} d\xi' d\rho'}{\rho'^2 + \xi'^2} \end{aligned} \quad (3.9 d)$$

(by substituting ξ' for $\xi' + 1$ and taking into account (3.9 b)).

The last member becomes identical with the expression for Q given in [(9.7)] if the constant $\chi_1 q_0 D$ is substituted for χ_1 (since $\chi_1 < 1$, and consequently $\sqrt{1 + \chi_1^2} \approx 1$). We can, therefore, directly write by [(20.5 b)] the following expression for $Q_{(12)2}$:

$$Q_{(12)2} = -\log \frac{\rho_1' \chi_1 q_0 D}{0.8}, \quad (3.9 e)$$

where, by (3.7) and (3.8), neglecting $Q_{(12)1}$,

$$\bar{W}_+ = -A^2 q_0 \left| \mp \log \frac{\rho_1' \chi_1 q_0 D}{0.8} \right|; \quad (3.10)$$

since $\rho_1' D = \rho_1$ is 0{1}* and $\chi_1 q_0 < 1$, the log will be negative, and as the expression in the || must be essentially positive (as, indeed, these brackets indicate), we must take $-$ in the double sign †, and consequently

$$\bar{W}_+ = A^2 q_0 \log \frac{\rho_1' \chi_1 q_0 D}{0.8}. \quad (3.10 a)$$

Hence, by [(19.1)], the distance force between two co-axial identical Whirls *when the adjustment of their relative phase angle is not interfered with by any external force* will be

$$F_{D+} = -\frac{A^2 q_0}{D}. \quad (3.11)$$

Thus two isolated co-axial identical Whirls are *mutually repelled* with a force *inversely proportional to the distance* (and *not* to the square of the distance).

* See p. 698 *loc. cit.* (text preceding formula [(20.5 c)]).

† i. e., $\cos \psi = +1$ (see (3.6 b) and (3.7)).

In applying this result to particles it will be more convenient to express this force in terms of their "electric charge" e .

From [(19.11 *i*)] we can find that $\frac{A^2}{e^2} = 1.97$. It must,

however, be recalled that the numerical value N_1^2 , on which this result is based, was obtained as an estimate depending, to a certain extent, on more or less arbitrary assumptions*. In connexion with the problems to which we have hitherto applied our theory, there was no opportunity to verify this numerical estimate by comparison with the observations. This estimate will, however, influence very essentially the theoretical value of the observable mass of the nuclei which we shall discuss in sections 5 and 7. It was found that much better agreement with observations would be obtained if we had $\frac{A^2}{e^2} = 1.79$ and *not* 1.97†. Therefore, by (3.10 *a*) and (3.11),

$$\bar{W}_+ = -1.79 q_0 e^2 \log \frac{\rho_1' \chi_1 q_0 D}{0.8}, \quad . \quad . \quad (3.12)$$

$$F_{D+} = -\frac{\pi^2}{2N_1^2} \frac{e^2 q_0}{D} = -1.79 q_0 \frac{e^2}{D}. \quad . \quad (3.12 a)$$

It will be recalled that this force is conditioned by $\psi = 0$, which corresponds to a stable equilibrium. If, however, owing to some external cause $\cos \psi$ will be different from unity, the force between the particles will be

$$F'_{D+} = F_{D+} \cos \psi. \quad . \quad . \quad . \quad (3.12 b)$$

In particular when, owing to an external cause, the Whirls rotate in the opposite directions, the average value of F'_{D+} will obviously be zero.

On the other hand, if, owing to an external cause, the phase difference between the Whirls is such that $\cos \psi < 0$, the force F'_{D+} will change its sign—that is to say, become a force of attraction instead of repulsion.

* See formula [(19.9 *g*)] and further.

† This will reduce by about 5 per cent. the force of electric and magnetic fields in the elementary particles, which is proportional to A (Phil. Mag. *loc. cit.* p. 700). The agreement of the now adopted figure with observations (see section 7) shows indirectly that our estimate of this field was of a correct order.

It follows from the above-stated assumptions that formulæ (3.10)–(3.12 *b*) are only applicable when $\chi_1 q_0 D \ll 1$.

We shall not attempt here to derive the exact formulæ (which really should be done by performing accurately the integration in (3.8 *c*), possibly with the help of (3.8 *d*)). We shall only state without proof that a result sufficiently accurate for our purpose can be attained by adding to our formulæ the factor $e^{-\chi_1 q_0 D}$. Then we shall have, instead of (3.12 *b*),

$$F'_{D+} = F_{D+} e^{-\chi_1 q_0 D} \cos \psi.$$

4. *Non-planetary Combination of Co-polarized Whirls (Sub-Ground State).*

In the expression for the distance force between two co-polarized Whirls which was derived in the above-cited paper in the Phil. Mag. (formula [(19.6)]) we have deliberately neglected terms of the $o\{(q_0 D)^{-1}\}$. Had we not neglected terms of this order we should obtain from [(19.5)] the following expression for this distance force instead of [(19.6)] :—

$$F_{D+} = \frac{\kappa_1 \kappa_2 \left(1 + \frac{q'_{01}}{q'_{02}}\right) C_2 \bar{N}_2^2}{D^2 2\sqrt{2}\pi} \times \left[\sqrt{2} \left(\sin\left(\varphi_1 \pm \frac{\pi}{4}\right) - \bar{f} \sin\left(\frac{\pi}{4} \pm \frac{\pi}{4}\right) \right) - \frac{2 \left(1 + \frac{q'_{01}}{q'_{02}}\right)}{q'_{01} D} \right] \quad (4.1)$$

From this, exactly in the same way as [(19.11)] was derived from [(19.6)], we should obtain, instead of [(19.11)], the following expression for the force of interaction between co-polarized particles of different signs,

$$F_{D+} = \frac{e^2}{D^2} \left(1 - \frac{2 \left(1 + \frac{q_{0-}}{q_{0+}}\right)}{q_{0-} D} \right) \cong \frac{e^2}{D^2} \left(1 - \frac{7.58 \times 10^{-11}}{D} \right), \quad (4.2)$$

by [(20.3)].

It follows from this that when the distance D between the co-polarized particles of different signs (say, proton and electron) is smaller than

$$D_0 = 7.58 \times 10^{-11} \text{ cm.}, \quad \dots \quad (4.3)$$

the force of their mutual attraction changes into a force of mutual repulsion.

From (4.2) we can easily find that at the distance

$$D = D_m = \frac{3}{2} D_0 = 1.137 \times 10^{-10} \text{ cm.} \quad (4.4)$$

this force reaches its maximum,

$$F_{D+} = F_{D_m+} \cong \frac{e^2}{3D_m^2} = 5.9 \text{ dyn.} \quad (4.5)$$

(+, -) (+, -)

By integration of (4.2) from $D = D_0$ to infinity we find the potential energy of a combination in equilibrium (i. e., when $D = D_0$),

$$E_D = \frac{e^2}{2D_0} = 1.5 \times 10^{-9} \text{ erg} = 950 \text{ el. volt.} \quad (4.6)$$

The gradient of force at the equilibrium point can be easily found from (4.2) and be expressed in the form*

$$\left(\frac{d}{d\delta} F_{D+} \right)_{\delta=\delta_0} = \frac{e^2 q_{0-}^2}{8 \left(1 + \frac{q_{0-}}{q_{0+}} \right)^4} \left(1 - \frac{2q_{0-}}{q_{0+}} \right) \cong \frac{e^2 q_{0-}^2 \hbar c}{8 \hbar c \left(1 + \frac{q_{0-}}{q_{0+}} \right)^6}$$

$$= 9.25 \times 10^{-4} q_{0-}^2 \hbar c, \quad (4.6 a)$$

where

$$\delta = q_0 - D, \quad \delta_0 = q_0 - D_0,$$

which corresponds to the wave-length $\lambda = 1.28 \times 10^{-8} \text{ cm.}$

Since D_0 is less than the distance corresponding to one quantum of rotational energy (ground state), it cannot correspond to any planetary orbit. At this distance the particles will only relatively oscillate about the equilibrium position.

We shall call such a combination of co-polarized particles "non-planetary" combination, or, perhaps still better, "sub-ground state."

A particle can also attach in the sub-ground state two co-polarized particles of the sign opposite to the former. Then, as is not difficult to see, the equilibrium distance, which we shall denote for this case by D_{10} , must satisfy the equation

$$e^2 \left(1 - \frac{D_0}{D_{10}} - \frac{1}{2} \right) = 0, \quad (4.7)$$

* Note that q_{0-} may vary during the motion, and, therefore,

$$\frac{d}{d\delta} = \frac{\partial}{\partial \delta} + \frac{\partial}{\partial q_{0-}} \frac{dq_{0-}}{d\delta} = \frac{\partial}{\partial \delta} + \frac{1}{D} \frac{\partial}{\partial q_{0-}}$$

and hence

$$D_{10} = \frac{4}{3} D_0 = 1.02 \times 10^{-10} \text{ cm.}, \quad . \quad . \quad (4.8)$$

$$E_{D_1} \cong 1.7 \times 10^{-9} \text{ erg.} \quad . \quad . \quad . \quad (4.9)$$

The gradient of force at the equilibrium can be easily derived by the same method as in the case of one particle in the sub-ground state, and can be written in the form

$$\left(\frac{d}{d\delta} F_{D+} \right)_{\delta=\delta_0} = 1.04 \times 10^{-3} q_0^2 \hbar c. \quad . \quad . \quad (4.10)$$

Apart from the "sub-ground" combination of two co-polarized Whirls, a system can be visualized in which both Whirls are concentric. Such a combination we can call "concentric."

It could be easily shown that if the co-polarized Whirls forming the concentric pair are also co-axial, their average mutual energy $\bar{W}$ (potential energy) must be zero. It follows also from symmetry considerations that a concentric state must represent an equilibrium.

We shall limit ourselves by pointing out the possibility of a "concentric state" in principle, but will not consider this state in detail in the present paper, as the previously obtained expressions for the force of interaction between the Whirls are not applicable to the case where the distance between their centres approaches zero.

5. Transformation of Energy, Mass, and Angular Momentum in a System of Whirls.

In the above-cited paper in the Phil. Mag. we discussed in detail the connection between the energy, inertial mass, and angular momentum of the Whirl at a given state of rest or of uniform motion. We found, on the basis of Maxwell Theory, that for any such given state of the Whirl the well-known Einstein relationships

$$E_i = mc^2 = \hbar_1 \Omega = \hbar_1 qc = \hbar_1 \beta q_0 c = m_0 \beta c^2,$$

where $\beta = (1 - v^2/c^2)^{-\frac{1}{2}}$, hold between the total electromagnetic energy of the Whirl E_i irrespective of its position ("internal" energy), its angular frequency $\Omega = 2\pi\nu$ (where ν is the number of cycles per second), the velocity v of the centre of the Whirl, its inertial mass m , the coefficient $q = \beta q_0$ representing the distance

frequency $\left(\frac{2\pi}{\lambda}, \text{ where } \lambda \text{ is the wave-length}\right)$ of the ordinary plane e.m. waves of the given frequency, and the coefficient $\hbar_1 = \frac{h_1}{2\pi}$ which represents one-half of the angular momentum of the Whirl, and is equal to the Planck constant $\hbar = \frac{h}{2\pi}$ when the Whirl is an elementary particle.

Although the connexion between different states of the same Whirl (like the constancy of the angular momentum), as well as the *tendencies* in the variation of the state, were fully discussed, the *process* of such a variation has only been outlined very briefly.

In connexion with the problem of the structure of nuclei it will be useful to elucidate these points a little further.

It can easily be seen from the expressions for the vector components of the compound Whirl*—and only such Whirls we shall here discuss—that to an infinitesimal variation of q_0 or v (and hence β) corresponds an infinitesimal variation of all the vectors and their first derivatives with respect to space coordinates and to time. In other words, the configurations of the Whirl, corresponding to the whole range of q_0 and v , form a *family* of solutions of the Maxwell equations, and to vary q_0 or v continuously means to *follow an envelope* of this family. Of course, at the contact points this envelope must have, at least, common *first* derivatives with the solutions with which it comes into contact. Since the Maxwell equations are linear, this means that the *envelope* will also satisfy the Maxwell equations—in fact it will form a so-called “singular solution.”

As is well known, the complete solution of the Maxwell equations will be fully determined by fixing the *initial* conditions (state of field throughout the space for the “initial” moment) and the *boundary* conditions (state of field at the boundary surface for any moment of time). This, however, can also be attained by fixing *only* the initial conditions, if in the latter the complete first-time derivatives of all the vector components are also included, which, in their turn, cannot be fully determined

* Formulæ [8.7]–[8.12].

unless $\frac{dv}{dt}$ and $\frac{dq}{dt}$ are also known. The latter two quantities are determined from the consideration of the "mutual" energy, from which the forces are defined, and from the constancy of the total energy*.

All these transformations are characterized by the constancy of the angular momentum† $2\hbar_1$, so that the internal energy of the Whirl will always be $\hbar_1\Omega$, and hence whenever this energy is converted into another form it will be $\hbar_1\Delta\Omega$, where $\Delta\Omega$ is the change of Ω .

Imagine two Whirls in a mobile equilibrium. This implies a periodic return of identical or nearly identical relative positions of both Whirls, which in its turn involves certain ratio of their frequencies. When, as a result of a disturbance involving the change of Ω of one of the Whirls by $\Delta\Omega$, a new stable position is attained, then the disturbed stability of the frequency relationship must be restored by means of another periodic motion (say, planetary motion) which we can call "stabilizing oscillation." If the frequency of this "stabilizing oscillation" is $\Omega_s = \frac{\Delta\Omega}{n}$ (where n is an integer), then the period of repetition of identical positions will be obviously increased n times, remaining unchanged when $n=1$.

In the case where the stabilizing oscillation is a planetary motion in the field of forces following the inverse square law, the total energy $\hbar_1\Delta\Omega$ will be equally divided—on the average over the whole cycle—between the potential and the kinetic energies, each being equal $\frac{1}{2}\hbar_1\Delta\Omega = \frac{1}{2}n\Omega_s$. Therefore, in this case the average angular momentum will be $n\hbar_1$. In the case of a different law of force a more detailed consideration may be required‡.

If a portion of the kinetic energy of the stabilizing oscillation will be converted into another form of energy (say of the precessional rotation of the axis of the Whirl), and the accompanying change of frequency will be $\delta\Delta\Omega$, then the transformed energy will be $\frac{\hbar_1}{2}\delta\Delta\Omega$. The frequency of the new oscillation, which we can call "stabilizing oscillation of the second order," must be, for reasons

* Phil. Mag. *loc. cit.* section 15.

† *Ibid.* section 6.

‡ See below, section 7.

stated above, $\Omega_s' = \frac{\delta \Delta \Omega}{m}$ (where m is an integer). Thus the angular momentum will be $\frac{m\hbar_1}{2}$. This will give rise to "half quantum numbers."

In more complex cases of disturbances accompanying the energy transformation the relationship may be more complex, but nevertheless they will have the same characteristic feature, namely, the appearance of "stabilizing oscillations" restoring the relative timing of periodic oscillations of the Whirls comprising the system as nearly as possible under given conditions. This recalls the Lenz law, according to which any disturbance in an electromagnetic system causes forces tending to resist this disturbance. The well-known quantum integers and simple fractions are only particular manifestations of this tendency towards the restoration of the relative timing.

It was recalled at the beginning of this section that, as a result of its classical electro-dynamical properties, the Whirl satisfies the relationship $E_i = mc^2 = \hbar_1 \Omega = \hbar_1 q c$. However, it should be borne in mind that this is the property of the Whirl as a whole, for the concept of the mass of the Whirl originated from the analysis of motion of its centre. The application of this concept to a volume element of the Whirl has no meaning.

The mass of the Whirl is essentially the manifestation of its internal energy E_i , which includes also the kinetic energy. If a portion of the internal energy is converted into the "mutual" energy W^* , the total mass of the system will be correspondingly reduced.

In a system of Whirls we shall call the sum of the masses of the component Whirls in an isolated state the "fundamental mass" of the system. Its actual mass we shall call "the resultant mass." The difference between the fundamental and the resultant masses—which must be due to the conversion of part of the internal energy into the mutual—we shall call "the mass deficiency," and denote it by m_s .

The value of m_s must depend on the actual circumstances under which the change of mass has occurred.

We shall consider here a few typical cases.

* Phil. Mag. *loc. cit.* p. 645.

Case I.

The particle moves from a point 0_1 , where the value of its mass is taken for a standard (fundamental mass). At a certain point 0_2 , as a result of a quick interaction ("collision") with another "material" particle or with a photon, it *acquires* an energy $\hbar \Delta \Omega$. The disturbance created by this can lead to a new stable state if it is compensated by a stabilising oscillation in the form of a planetary motion with the angular velocity $\Omega_s = \frac{n_1}{m} \Delta \Omega$ and an accompanying increase of the rest frequency of the particle $\frac{n_2}{m} \Delta \Omega$, where n_1, n_2 , and m are whole numbers. Then the energy $\hbar \frac{m-n_2}{m} \Delta \Omega$ will be transformed into the energy of the planetary motion, which will be the kinetic energy K_s , since the process is assumed to last too short a time for any appreciable change in the potential energy. Thus

$$\left. \begin{aligned} K_s &= \hbar \frac{m-n_2}{m} \Delta \Omega = \hbar \frac{m-n_2}{n_1} \Omega_s. \\ m &> n_2 \geq 1. \end{aligned} \right\} \quad . \quad . \quad (5.1)$$

Let W_s be the difference between the average potential energy of the particle at the planetary orbit and its potential energy at a point 0_1 (say very remote point), where its mass is taken for a standard. Then the loss of mass will correspond to the energy

$$W_s - \hbar \Delta \Omega = W_s - \hbar \frac{m}{n_1} \Omega_s. \quad . \quad . \quad . \quad (5.2)$$

The loss of mass in conventional units (in which the mass of the proton is taken as 1.0081) will be

$$\left. \begin{aligned} m_s &= \frac{1.0081}{\hbar q_0 + c} \left(W_s - \hbar \frac{m}{n_1} \Omega_s \right) = \frac{1.0081}{\hbar q_0 + c} \left(W_s - \frac{m}{m-n_2} K_s \right). \\ m &> n_1 \geq 1, \\ m &> n_2 \geq 1. \end{aligned} \right\} \quad . \quad . \quad . \quad (5.3)$$

The disturbance will be minimum if $n_1 = n_2 = 1$, and $m = 2$. Then

$$\left. \begin{aligned} K_s &= \hbar \Omega_s, \\ m_s &= \frac{1.0081}{\hbar q_0 + c} (W_s - 2\hbar \Omega_s) = \frac{1.0081}{\hbar q_0 + c} (W_s - 2K_s) \end{aligned} \right\}. \quad (5.3a)$$

Case II.

Assume now that the energy is not *acquired* from the outside source, but is *lost* to it.

Then the results of Case I. can be applied if, taking into account that now $\Delta\Omega < 0$ while essentially $\Omega_s > 0$

and $\frac{n_2}{m} \Delta\Omega < 0$, we shall change the signs before m and n_2 .

Then we shall have

$$\left. \begin{aligned} K_s &= \hbar \frac{n_2 - m}{n_1} \Omega_s, \\ m_s &= \frac{1.0081}{\hbar q_0 + c} \left(W_s + \hbar \frac{m}{n_1} \Omega_s \right) = \frac{1.0081}{\hbar q_0 + c} \left(W_s + \frac{m}{n_2 - m} K_s \right). \end{aligned} \right\} \quad \begin{aligned} n_2 &> m \geq 1, \\ n_1 &\geq 1. \end{aligned} \quad (5.4)$$

The disturbance will be minimum when $m = 1$, $n_2 = 2$. Then

$$\left. \begin{aligned} K_s &= \hbar \Omega_s, \\ m_s &= \frac{1.0081}{\hbar q_0 + c} (W_s + \hbar \Omega_s) = \frac{1.0081}{\hbar q_0 + c} (W_s + K_s). \end{aligned} \right\}. \quad (5.4a)$$

Case III.

Let the process be similar to Case II., only the disturbance started when the mass was very near to the "fundamental mass," and it lasted a sufficiently long time for the completion of the change of the potential energy by W_s . Then, as is not difficult to see,

$$\left. \begin{aligned} K_s + W_s &= \hbar \frac{n_2 - m}{n_1} \Omega_s, \\ m_s &= \frac{1.0081}{\hbar q_0 + c} \hbar \frac{m}{n_1} \Omega_s = \frac{1.0081}{\hbar q_0 + c} \frac{m}{n_2 - m} (W_s + K_s). \end{aligned} \right\} \quad \begin{aligned} n_2 &> m \geq 1. \end{aligned} \quad (5.5)$$

In the simplest case, $m=1$, $n_2=2$,

$$\left. \begin{aligned} K_s + W_s &= \hbar \Omega_s, \\ m_s &= \frac{1.0081}{\hbar q_{0+c}} \hbar \Omega_s = \frac{1.0081}{\hbar q_{0+c}} (W_s + K_s). \end{aligned} \right\} \quad (5.5 a)$$

When the inverse square law holds, $K_s=W_s$, and hence $K_s = \frac{n_2 - m}{n_1} \frac{\hbar \Omega_s}{2}$, and consequently the angular momentum of the planetary motion equals $\frac{n_2 - m}{n_1} \hbar$. Its minimum will be $\hbar$ (as compared with the angular momentum $2\hbar$ of the particle). This minimum angular momentum will, for example, correspond to the "ground state" of the hydrogen atom.

Case IV.

Consider now the case where a particle is in a state of oscillation about its equilibrium position.

In this oscillation the potential and the kinetic energy will be mutually converted in the alternating direction with the frequency twice as high as the frequency of oscillations, but there will be no external loss of energy.

The loss of mass due to this oscillation will correspond to the average potential energy W_s during this oscillation. Since the potential energy W_s is essentially a function of the position x , its averaging must *not* be done with respect to time t but with respect to the position x .

Consider now a simple harmonic oscillation. To find the coefficient of averaging $\frac{W_s}{W_{s \max.}}$, where $W_{s \max.}$ is the amplitude of W_s , we can, without loss of generality, assume that the amplitude of x , and the angular frequency Ω_s of the oscillations both equal unity. Then

$$\left. \begin{aligned} x &= \sin t, \\ W_s &= W_{s \max.} \cos^2 t = \frac{W_{s \max.}}{2} (1 + \cos 2t), \\ \bar{W}_s &= W_{s \max.} \int_0^1 \cos^2 t \, dx = W_{s \max.} \int_{\frac{\pi}{2}}^0 \cos^2 t \, dt = \frac{2}{3} W_{s \max.} \end{aligned} \right\} \quad (5.6)$$

Consider now the relationship between the angular frequency Ω_s of the oscillations of the particle and the

maximum variation $\Delta\Omega$ of the angular velocity Ω of its rotation about its own axis of symmetry.

In order that an oscillation should keep pace with a continuous rotation, the *average* velocity of the former must be equal to the unidirectional velocity of the latter. Therefore we must have

$$\frac{2}{\pi} \Omega_s = \Delta\Omega. \quad . \quad . \quad . \quad . \quad (5.6 a)$$

From the conservation of the total energy we have

$$W_{s \max.} = \hbar \Delta\Omega, \quad . \quad . \quad . \quad . \quad (5.6 b)$$

and hence

$$\frac{m_s}{1.0081} = \frac{\bar{W}_s}{\hbar q_{0+} c} = \frac{2W_{s \max.}}{3\hbar q_{0+} c} = \frac{2}{3q_{0+} c} \Delta\Omega = \frac{4}{3\pi} \frac{\Omega_s}{q_{0+} c}. \quad (5.7)$$

Let F_D be the force of attraction of the particle, and the positive direction of the distance D be counted in the usual way from the centre of attraction.

Then taking the oscillations as approximately simple harmonic we can easily find

$$\Omega_s = \sqrt{\frac{1}{m} \left(\frac{dF_D}{dD} \right)_{D=D_0}}, \quad . \quad . \quad . \quad . \quad (5.8)$$

where m is the mass of the particle and D_0 the value of D at the equilibrium.

Hence, recalling that $mc^2 = \hbar qc$ (where $q = \frac{\Omega}{c}$), we find by (5.7)

$$\begin{aligned} m_s &= \frac{1.0081}{q_{0+}} \frac{4}{3\pi} \sqrt{\frac{1}{q\hbar c} \left(\frac{dF_D}{dD} \right)_{D=D_0}} \\ &= 1.0081 \frac{q}{q_{0+}} \frac{4}{3\pi} \sqrt{\frac{1}{q^2\hbar c} \left(\frac{dF_D}{d\delta} \right)_{\delta=\delta_0}}, \quad (5.9) \end{aligned}$$

where

$$\delta = qD, \quad \delta_0 = qD_0.$$

However, the coefficient $\frac{4}{3\pi}$ in (5.7) and (5.9) is applicable only in those cases where the potential energy $\bar{W}$ can be regarded as an aperiodic function of D or, at least, when the period of this function is small in comparison with the total displacement.

In those cases where it is a periodic function and the displacement is of the same order as the period, the oscillation will take place within the same period of potential energy (or, perhaps, with an occasional "jump" from one period to another, which is really not essential), and then the oscillation must be regarded as an *oscillation of phase* of W , or an oscillation of $\sin \Phi$ (see, for example, formulæ (2.8) and (2.9)), which in its turn is caused by the oscillation of $q_0 D$ (see formulæ [(19.9 c)] (2.6) and [(19.10 e)]). This will alter the relationship between Ω_s and $\Delta\Omega$, on the one hand, and between the average and the maximum values of the potential energy during the oscillation, on the other.

It was previously shown *: (a) that the oscillations of $q_0 D$ will balance the unidirectional change of the other component of Φ , the $(q_{01} \pm q_{02})ct$, if the total period of oscillation is *equal* to the period of the $(q_{01} \pm q_{02})ct$, and (b) that, as a result, the average value of $\sin \Phi$ will be about *one-half* ($\epsilon'^2/2 \cong 1/2$). Consequently, we must put

$$\Delta\Omega = \Omega_s \quad \text{and} \quad \frac{W_s}{W_{s, \max.}} = \frac{1}{2}.$$

Therefore, when the potential energy is a periodic function of D , and when the amplitude of the stabilizing oscillation is comparable with the period of this function, we must substitute the coefficient $\frac{1}{2}$ for $\frac{4}{3\pi}$ in formulæ (5.7) and (5.9).

As an example of an *aperiodic* potential energy we can take the potential energy between two co-axially situated copolarized Whirls of equal size. In the other examples which we have hitherto considered the potential energy is a *periodic* function of the distance D .

Hitherto we assumed that only *one* of the particles is moving and alters its Ω as a result of a disturbance. Of course, in reality this is not so, for the rest of the system will recoil and will also alter the Ω of its components.

The influence of this fact on m_s depends on the actual circumstances. When the recoiling part is very heavy in comparison with the part which was taken as "oscillating," the influence of the recoiling part on m_s will be

* Phil. Mag. *loc. cit.*, the text between formulæ [(19.10 a)] and [(19.10 d)].

very small. When, on the other hand, the mass of the recoiling part is equal to that of the "oscillating," the m_s must be obviously doubled, as in such a combination the total amplitude of D and the total maximum of the potential energy will be also doubled. As an example of such a case, one can take the oscillation of two contrapolarized particles of equal size, which was discussed in section 2.

The m_s can be essentially affected not only by the individual properties of the oscillating particles, but also by a symmetry which may result from their orientation in relation to other particles. If, for example, two identical and mutually oscillating particles are orientated anti-symmetrically (for example, when $\psi = \pi$ in the case discussed in section 3), they will form a system with a complete period corresponding to one-half of the revolution.

Consequently $\Delta\Omega$, and hence, for this reason, m_s , must be halved on this account. On the other hand, as has been just said, the equality of the two mutually oscillating parts must lead to the doubling of m_s . Therefore, as a total result the $\frac{4}{3\pi}$ in formulæ (5.7) and (5.9) will remain unaltered.

6. *The Structure of the Neutron (${}_0n^1$), the Deuteron (${}_1H^2$), the Nucleus ${}_1H^3$, the Nucleus ${}_2He^3$, and the α -particle (${}_2He^4$)*.*

In discussing the structure of nuclei we shall first visualize this structure and then try to show that it agrees with the theory.

This structure can be best represented graphically. We shall use a simplified graphic representation in which any elementary particle is represented by a co-axial and concentric flat disk. Positive particles (*i. e.*, protons and magneto-protons) will be represented by smaller disks drawn in heavier lines, while negative particles (*i. e.*, electrons and magneto-electrons or positrons) will be represented by larger disks drawn in thinner lines. Disks representing the electropolarized particles will be drawn in solid lines, while disks representing the magnetopolarized particles (*i. e.*, magneto-protons and

* Hydrogen isotopes are denoted by H with the appropriate suffixes. Special letters, like D for Deuteron etc., are not used here.

magneto-electrons or positrons) will be drawn in dotted lines.

Of course these disks will not be drawn to scale, for if they were the disks representing the negative particles would have to be about 1840 times as large as the disks representing the positive ones.

Then the structure of the neutron (${}_0n^1$), deuton (${}_1H^2$), the nucleus ${}_1H^3$, the nucleus ${}_2He^3$, and the α -particle (${}_2He^4$) will be represented in figs. 2 (a), 2 (b), 3 (a), 3 (b), and 4 resp.

We shall assume in this section that the component particles forming the nuclei have a fixed position apart from very rapid oscillations of the "vibrational state" *.

In other words, subject to the reservations implied in the vibrational state, we shall consider in this section the nuclei in *static* equilibrium. The *dynamic* equilibrium will be described in the next section in conjunction with the "mass effect."

It was previously suggested that the neutron ${}_0n^1$ is a combination similar to the hydrogen atom, only with magneto-polarized components. Here we shall modify this suggestion by assuming that, generally speaking, the magneto-proton and the magneto-electron (positron) in the neutron are in *sub-ground state*. This may better account for the stability of the neutron.

It should be noted that, owing to the smallness of the force of interaction between the contrapolarized particles (unless they are of the same sign and situated in close proximity †), free neutrons should be practically indistinguishable from the free magneto-protons.

As aforesaid (section 4), the positive particle can attach one or two negative particles in the sub-ground state. It will be more convenient to postpone until the next section the consideration of the question whether the neutron in a particular nucleus contains one or two magneto-electrons (positrons), and in our graphical representation we shall show it with only one magneto-electron.

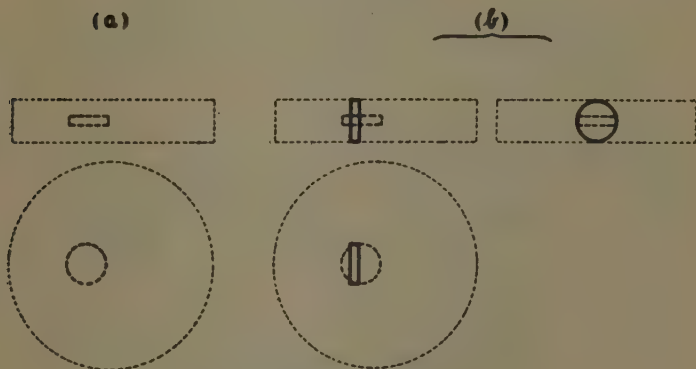
The deuton ${}_1H^2$ (fig. 2 (b)) consists of a proton and a neutron. In accordance with the theory, the axis of the proton is perpendicular to the axis of the magneto-proton (in the neutron).

* See Phil. Mag. loc. cit. p. 670.

† Loc. cit. pp. 693, 699, and 700.

The presence of the positron in the neutron will create a tendency for the axis of the *proton* to coincide with the axis of symmetry of the deuteron as a whole, for otherwise (*i. e.*, if the axis of symmetry of the deuteron coincided with that of the *magneto-proton* in the neutron) the central planes of both members of the neutron (the magneto-proton and the positron) would have a tendency to separate, and thus upset the symmetry of the whole system, for unless the central planes of both members of the neutron coincide and include the axis of the proton their natural distances from the latter will be different*.

Fig. 2.



Diagrams showing the structure of nuclei.

(a) Neutron (${}_0^1\text{n}$).(b) Deuteron (${}_1^2\text{H}$).

Smaller disks represent protons when drawn in solid lines and magneto-protons when in dotted. Larger disks represent electrons when in solid lines and magneto-electrons (positrons) when in dotted.

Since the force of interaction between the positron and the proton will be comparatively small, we can use for determining the distance between the centres of the proton and magneto-proton the result obtained in (2.12b), that is to say, take for this distance 1.24×10^{-13} cm.

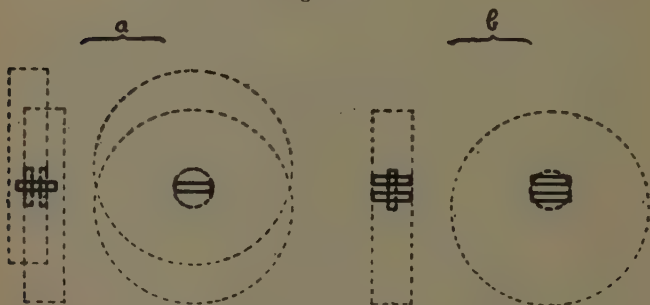
Also we can use the first formulæ (2.12) and (2.12f) to calculate the force and the energy required for separating a proton from the deuteron.

* Compare (2.12) and [(20.7)].

The nucleus ${}_1\text{H}^3$ (fig. 3 (a)) consists of a proton and two neutrons. The proton is the centre of symmetry. As has been already mentioned, the central planes of both members of each neutron will have a tendency to become mutually displaced, but in this case such displacement will not upset the symmetry of the system as a whole.

Regarding the relative rotation of the oppositely situated magneto-protons about their common axis, there are two possibilities: the one that they will rotate in opposite directions, the other that they will rotate in the same direction. In the first case the average force between the protons will be zero, in the second the sign of the force depends on their phase difference (ψ).

Fig. 3.



Diagrams showing the structure of nuclei.

- (a) The nucleus ${}_1\text{H}^3$.
 (b) The nucleus ${}_2\text{He}^3$.

It can be easily observed that the phase relationship of each of the two opposite magneto-protons to the proton should correspond to $\psi = \pi$, for, on the average, the phases of the proton in symmetrical points are diametrically opposite. Then $\cos \psi = -1$, and we shall have attraction between the magneto-protons instead of repulsion, as would be the case in an isolated system of two co-axial identical Whirls.

We shall first consider the second possibility. We shall denote here by D_1 the distance of each magneto-proton from the centre of the nucleus (*i. e.*, the distance between the magneto-protons will be $2D_1$).

By (2.12 *a*), (3.12), and (3.12 *a*) we find that the expression for the total force F_{D_1} acting on a magneto-proton (attraction to the opposite magneto-proton and to the proton) can be written in the following form :

$$\begin{aligned} F_{D_1} &= e^2 \left(\frac{1.79 q_{0+}}{2 D_1} + \frac{4}{\pi q_{0+} D_1^3} - \frac{24}{\pi q_{0+}^2 D_1^4} \right) \\ &= \frac{1.79 e^2 q_{0+}^2}{2} \left(\frac{1}{q_{0+} D_1} + \frac{8}{1.79 \pi (q_{0+} D_1)^3} - \frac{48}{1.79 \pi (q_{0+} D_1)^4} \right) \\ &= 4.8 \times 10^8 \left(\frac{1}{q_{0+} D_1} + \frac{1.43}{(q_{0+} D_1)^3} - \frac{8.55}{(q_{0+} D_1)^4} \right), \quad (6.1) \end{aligned}$$

by [(20.2)] and [(20.3)].

The position of equilibrium ($F_{D_1}=0$) will correspond to the real root of the trinomial in brackets, namely, to

$$q_{0+} D_1 = 1.81$$

or

$$D_1 = 3.74 \times 10^{-14} \text{ cm.} \quad (6.2)$$

By (6.1) the gradient of force at the equilibrium point can be expressed as follows :—

$$\begin{aligned} \left(\frac{dF_{D_1}}{d\delta} \right)_{\delta=\delta_0} &= - \frac{1.79 e^2 q_{0+}^2}{2} \left(\frac{1}{\delta_0^4} + \frac{4.29}{\delta_0^4} - \frac{34.2}{\delta_0^5} \right) \\ &= 0.01173 q_{0+}^2 \hbar c, \quad (6.2 a) \end{aligned}$$

where $\delta = q_{0+} D_1$, $\delta_0 = q_{0+} D_{10} = 1.81$.

The form of expression given in the third member will be useful in the future.

The value of D_1 corresponding to the maximum force F_{D_1} holding the magneto-proton in position will be determined by the condition $\frac{\partial F_{D_1}}{\partial D_1} = 0$, which, by (6.1), is equivalent to the condition

$$(q_{0+} D_1)^3 + 4.29 q_{0+} D_1 - 34.2 = 0, \quad (6.3)$$

whence, denoting this value of D_1 by D_{1m} , we have

$$q_{0+} D_{1m} = 2.81. \quad (6.3 a)$$

Hence by (6.1) the maximum holding force of the magneto-protons

$$F_{D_{1m}} = 1.36 \times 10^8 \text{ dyn.} \quad (6.3 b)$$

The energy corresponding to the force of attraction between the co-axial magneto-protons, which corresponds to the first term of (6.1), will be found from (3.12). The energy corresponding to the force of interaction between the magneto-protons and the proton will be found from the second and third terms of (6.1) by integration and doubling the result.

Hence we can write for this energy

$$\begin{aligned} E_{D_1} &= \frac{4.8 \times 10^8}{q_{0+}} \left[-2 \log (2\chi_1 q_{0+} D_{10}) \right. \\ &\quad \left. + 2 \int_{q_{0+} D_{10}}^{\infty} \left(\frac{1.43}{(q_{0+} D_1)^3} - \frac{8.55}{(q_{0+} D_1)^4} \right) d(q_{0+} D_1) \right] \\ &\cong 10^{-5} \left(-2 \log (10^{-31} \times 3.62) + \frac{1.43 \times 2}{2 \times 1.81^2} - \frac{8.55 \times 2}{3 \times 1.81^3} \right) \\ &\cong 1.4 \times 10^{-3} \text{ erg} \cong 8.8 \times 10^8 \text{ el. volt.} \quad . \quad . \quad . \quad (6.4) \end{aligned}$$

This corresponds to the wave-length

$$\lambda_{D_1} \cong 1.45 \times 10^{-13} \text{ cm.} \quad . \quad . \quad . \quad (6.4a)$$

We can thus see that this energy is comparable with the internal energy of particles.

However, this energy cannot be regarded as the *minimum* energy required for the destruction of the nucleus, for the phase difference $\psi = \pi$ is maintained only by the interaction between the magneto-protons and protons. As soon as the distance D_1 becomes sufficiently large (say of the order of 10^{-10} cm.) this interaction becomes weak, and the phase difference $\psi = \pi$ (which, as will be recalled, corresponds to the unstable equilibrium in the case of an isolated pair of co-axial magneto-protons) becomes comparatively unstable. Thus the separation energy depends on the character of the disturbing force.

If this force is co-axial with the magneto-protons the separation energy is very large, but it can be much smaller if simultaneously with this force a torque about the axis of the magneto-proton is applied which is capable of altering the phase difference*.

Consider now the case where the opposite co-axial magneto-protons rotate in opposite directions. In that case the average force between the opposite magneto-

* The law of conservation of energy still holds, as the sum of the mutual and the internal energies remains constant.

protons will be zero, and the force and the energy can be determined from (2.12 *d*) and (2.12 *f*) resp.

The force and the energy will be much smaller, and, moreover, there will be moments (at $\cos \psi < 0$) where the total momentary force acting on magneto-protons will be repelling instead of attracting.

The nucleus ${}_2\text{He}^3$ (fig. 3 (*b*)) consists of two protons and one neutron. Since the forces between the protons and positrons (entering into neutrons) are comparatively negligible, this nucleus will not differ essentially as regards the distances and holding forces from the nucleus ${}_1\text{H}^3$. Only the protons in the former will play the same part as the magneto-protons in the latter.

The α -particle ${}_2\text{He}^4$ (fig. 4) consists of two mutually perpendicular pairs of protons and neutrons. The forces

Fig. 4.

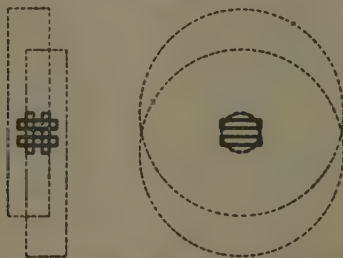


Diagram showing the structure of the α -particle (${}_2\text{He}^4$).

of interaction can be considered exactly in the same manner as in the case of ${}_1\text{H}^3$ or ${}_2\text{He}^3$.

The distance force $F_{D,1}$ between the oppositely situated protons and magneto-protons in the α -particle can be calculated exactly by the same method as was used in the case of ${}_1\text{H}^3$, with the sole difference that the last two terms in () in the equation (6.1) must be doubled, owing to the fact that in the α -particle each proton or magneto-proton will be acted upon not by one contrapolarized particle of the same sign as in ${}_1\text{H}^3$ or ${}_2\text{He}^3$ but by two.

Again, by the same method as in the case of H_1^3 we shall find for the equilibrium distance D_{10}

$$q_0 + D_{10} = 2.21,$$

whence

$$D_{10} = 4.55 \times 10^{-14} \text{ cm.}, \quad . \quad . \quad . \quad (6.5)$$

and for the distance D_{1m} corresponding to the maximum force

$$q_0 + D_{1m} = 3.4, \quad . \quad . \quad . \quad (6.5 a)$$

and the maximum distance force

$$F_{D_{1m}} = 1.28 \times 10^8 \text{ dyn.}, \quad . \quad . \quad . \quad (6.5 b)$$

and the maximum energy

$$E_{D_1} \cong 2.8 \times 10^{-3} \text{ erg} \cong 1.76 \times 10^9 \text{ el. volt.}, \quad (6.5 c)$$

which corresponds to the wave-length

$$\lambda_{D_1} \cong 7.3 \times 10^{-14} \text{ cm.}, \quad . \quad . \quad . \quad (6.5 d)$$

Fig. 5.

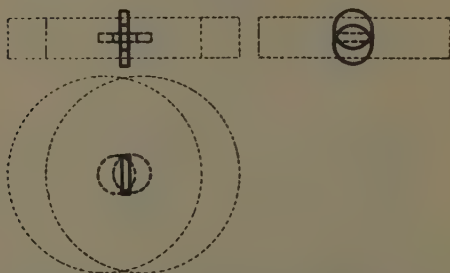


Diagram showing an alternative structure of α -particle, which, however, cannot be balanced.

and the gradient of force at the equilibrium point can be written in the form

$$\left(\frac{dF_{D_1}}{d\delta} \right)_{\delta=\delta_1} = 0.004867 q_0^2 \hbar c, \quad . \quad . \quad (6.5 e)$$

where

$$\delta = q_0 + D_1, \quad \delta_0 = q_0 + D_{10}.$$

An alternative picture of the α -particle, which is shown in fig. 5, may also be regarded as possible. It is, however, easy to find from [(19.11)] and (2.12 d) that such a system cannot be stable, as the repelling forces will be stronger than the attracting.

7. The Resultant Masses.

First of all we shall determine the masses of the neutron and of the nuclei considered in the previous section, *without* taking into account the loss of mass due to the relative motion of the component particles, or, as we shall call it, the "fundamental masses." This can be done directly from the data given in the previous section as soon as it is determined which type of neutron will enter into the system, the one with one magneto-electron or the one with two symmetrically situated magneto-electrons.

A neutron with only one magneto-electron will, generally speaking, have a very great chance of coming into proximity with a second free magneto-electron and capturing it. Therefore we must assume that a neutron has *two* magneto-electrons, unless there is a cause preventing their co-existence. It can be easily seen that such a cause exists only in ${}_1\text{H}^3$ and ${}_2\text{He}^4$, where, owing to the mutual repulsion of the magneto-electrons, only a single magneto-electron can be present in each of the neutrons, both magneto-electrons being diametrically opposite (or nearly so).

We shall denote the neutron with only one magneto-electron (a "defective neutron") by the symbol ${}_0\bar{n}^1$.

So we can easily find :

TABLE I.
The Fundamental Mass.

	Calculation.	Result.
${}_1\text{H}^1$	—	1.0081
${}_0\bar{n}^1$	$1.0081 + 0.00055$	1.00865
${}_0n^1$	$1.0081 + 2 \times 0.00055$	1.0092
${}_1\text{H}^2$	$1.0081 + 1.0092$	2.0173
${}_1\text{H}^3$	$1.0081 + 2 \times 1.00865$	3.0254
${}_2\text{He}^3$	$2 \times 1.0081 + 1.0092$	3.0254
${}_2\text{He}^4$	$2 \times (1.0081 + 1.00865)$	4.0335

The method of determining the difference between the fundamental mass and the resultant mass ("the mass deficiency") was described above (section 5).

The actual way in which this method will be applied will depend on the character of the stabilizing oscillations in a given system.

In both kinds of neutrons, ${}_0\bar{n}^1$ and ${}_0n^1$, there must be *linear* oscillations, in accordance with our fundamental hypothesis about their structure.

The mass deficiency due to these oscillations can be easily found from (5.9) on introducing in it

$$\frac{q}{q_{0+}} = \frac{q_{0-}}{q_{0+}} = \frac{m_{0-}}{m_{0+}} = \frac{1}{1846},$$

substituting the coefficients $\frac{1}{2}$ and $2 \times \frac{1}{2} = 1$ for $4/3\pi$, in cases of ${}_0\bar{n}^1$ and ${}_0n^1$ resp. (see the end of section 5), and,

finally, introducing the values for $\left(\frac{dF_D}{dD}\right)_{D=D_0}$ from (4.6 a)

and (4.10), for ${}_0\bar{n}^1$ and ${}_0n^1$ resp.

Taking the mass at a very remote point as a standard mass, we must subtract from the obtained results the masses corresponding to the potential energies of the equilibrium positions. These will be found by dividing

into $\frac{\hbar q_{0+}c}{1.0081} \simeq 1.25 \times 10^{-3}$ the potential energies of the

equilibrium positions, drawn from formulæ (4.6) and (4.9) in the case of ${}_0\bar{n}^1$ and ${}_0n^1$ resp.*

In the nucleus ${}_1H^2$ we must also assume *linear* oscillations, and not a planetary motion, simply because the attracting force is insufficient to balance the centrifugal force even at a minimum angular momentum which would assure the restoration of timing referred to in section 5, or, in other words, which would satisfy the quantum relationships. This would not be difficult to find.

The mass deficiency corresponding to these oscillations will be found from (5.9) on introducing into it the value

of $\left(\frac{dF_D}{dD}\right)_{D=D_0}$ from (2.12 e)† and substituting 1 for the co-

efficient $4/3\pi$, in accordance with the observations at the end of section 6. To this must be added the mass corresponding to the potential energy of equilibrium

* The results are given in Table II. (p. 579).

† Since the interaction between the proton and the magnteo-electron is negligible.

point by dividing the value of this energy given in (2.12 f) into $\frac{\hbar q_0 + c}{1.0081} \cong 1.25 \times 10^{-3}$.

Let us now consider the ${}_1\text{H}^3$ and ${}_2\text{He}^3$. In these particles the planetary motion of the diametrically opposite particles (magneto-protons in ${}_1\text{H}^3$ and protons in ${}_2\text{He}^3$) is possible. It is not difficult to see that, owing to the rotation of the diametrically opposite particles in the same direction round their axis of symmetry, this planetary motion is not likely to influence the rotation of the middle particle (the proton in ${}_1\text{H}^3$ and the magneto-proton in ${}_2\text{He}^3$) about its own axis, for the directions in which the middle particle will tend to be rotated by the diametrically opposite ones will be obviously opposite, at least on the average. Therefore, following our Minimum Disturbance Principle *, we must regard the planetary motion as more likely.

For further analysis it will be necessary to investigate a little closer the motion of the diametrically opposite particles.

In the previous section the expression of the mutual energy was derived on the assumption $\psi = \pi$ †. For a more general case, assuming that the motion of both particles is symmetrical, the expression for W can be written as follows :

$$\bar{W} = 1.79 \frac{e^2}{\hbar c} \left[-\log (2\chi_1 \delta) \cdot \cos \psi + \left(\frac{0.715}{\delta^2} - \frac{2.83}{\delta^3} \right) \frac{1+a \cos \psi}{1+a} \right], \quad (7.1)$$

where $\delta = qD$ and a is a factor smaller than unity. Here the coefficient $\frac{1+a \cos \psi}{1+a}$ is an artificial factor equal to

unity at $\psi = \pi$ which was introduced to express the tendency of forces between the proton and the magneto-proton in the nuclei under consideration to keep $\psi = \pi$ (for $d^2/d\psi^2$ of this coefficient at $\psi = \pi$ is obviously positive) ‡. Apart from this artificial coefficient, the expression (7.1) can be easily derived in the same way as formula (6.4), by taking into account (3.12 b).

* Phil. Mag. loc. cit. section 18, p. 675.

† Formula (6.4).

‡ See the text preceding formula (5.1).

Recalling that the force is equal $-\frac{1}{2} \frac{dW}{dD} = -\frac{q}{2} \frac{d\bar{W}}{d\delta}$ and the torque tending to alter ψ equals $-\frac{d\bar{W}}{d\psi}$, and also that the mass equals $\frac{h q}{c}$, while the moment of inertia equals $\frac{2\hbar}{qc}$ (the angular momentum over the angular velocity), we can easily obtain from (7.1) the following equations of motion of the diametrically opposite particles:

$$\left. \begin{aligned} \frac{d^2\delta}{d\tau^2} &= \frac{1.79e^2}{2\hbar c} \left[-\frac{\cos\psi}{\delta} + \left(\frac{1.43}{\delta^3} - \frac{8.55}{\delta^4} \right) \frac{1+a\cos\psi}{1-a} \right], \\ \frac{d^2\psi}{d\tau^2} &= \frac{1.79e^2}{2\hbar c} \left[\log(2\chi_1 \delta) + \left(\frac{0.715}{\delta^2} - \frac{2.83}{\delta^3} \right) \frac{a}{1-a} \right] \sin\psi, \end{aligned} \right\} \quad \dots (7.2)$$

where $\tau = cqt$ is time in "invariant" units. These equations of motion are independent of q , and thus of the variation of the mass.

Recalling that χ_1 is of the order 10^{-31} *, we can see by a glance at these equations that the general character of relative motion of diametrically opposite particles will be a comparatively fast relative rotation about their axis of symmetry with fluctuating, or possibly alternating, angular velocity, and a comparatively slow axial motion with alternating direction of speed †. If during this motion there be any disturbance leading to the stabilizing planetary motion ‡ with a radius of orbit δ_s , then, of course, in the latter the centrifugal acceleration $\S \delta_s \left(\frac{\Omega_s}{cq} \right)^2$ and the centripetal $\left(\frac{d^2\delta}{d\tau^2} \right)_{\delta=\delta_s}$ must be balanced.

The kinetic energy of this motion,

$$K_s = 2 \times \frac{1}{2} m \left(\frac{\delta_s}{q} \right)^2 \Omega_s^2,$$

* See formula [(20.5 c)].

† We shall limit ourselves to this observation on the general character of motion, without attempting to solve the differential equations (7.2), as this would be superfluous for our immediate object. It may, however, be noted that the solution of equations (7.2) does not present any insuperable difficulties.

‡ See section 5, "Case I."

§ Using the same units of time as in (7.2).

will be supplied by the disturbed particles on account of the decrease of their internal energies, which is due to the reduction of the sum of their Ω by $\Delta\Omega=\Omega_s$, and, therefore, will be equal to $\hbar\Delta\Omega=\hbar\Omega_s$.

Consequently,

$$\delta_s \left(\frac{\Omega_s}{cq} \right)^2 = \left(\frac{d^2\delta}{d\tau^2} \right)_{\delta=\delta_s} \quad . \quad . \quad . \quad (7.3)$$

and (recalling that $m=\hbar q/c$) the kinetic energy

$$K_s = \frac{\hbar \delta_s^2 \Omega_s^2}{cq} = \hbar \Omega_s. \quad . \quad . \quad . \quad (7.3 a)$$

By (7.3) and (7.3 a)

$$\frac{K_s}{\hbar qc} = \frac{\Omega_s}{qc} = \delta_s^{-2}, \quad . \quad . \quad . \quad . \quad (7.4)$$

$$\delta_s^3 \left(\frac{d^2\delta}{d\tau^2} \right)_{\delta=\delta_s} = 1. \quad . \quad . \quad . \quad . \quad (7.4 a)$$

To maintain the stability of the attracting force in the planetary motion there must be $\psi \cong \pi$. Hence by (7.2) and (7.4 a)

$$\frac{1.79e^2}{2\hbar c} \left(\delta_s^2 + 1.43 - \frac{8.55}{\delta_s} \right) = 1, \quad . \quad . \quad . \quad (7.5)$$

whence, on introducing the values from [(20.1)] and [(20.2)] and solving this equation,

$$\delta_s = 12.36. \quad . \quad . \quad . \quad . \quad (7.5 a)$$

It remains for us to determine the potential energy W_s .

From (7.1) we can easily find the difference of the potential energy at $\delta=\delta_s=12.36$ (by (7.5 a)) and at the static equilibrium, i. e., $\delta=\delta_0=1.81$ (by (6.2)); in both cases $\psi=\pi$.

The question is now how the W at the static equilibrium compares with the W when the particles are isolated. We can see from (7.1) that this really depends upon the conditions at the moment when the two diametrically opposite particles became co-axial (at a large δ , and in particular on the $\cos\psi$ at that moment *). Depending on this the mass may increase or decrease with

* Compare with the observations following formula (6.4 a).

nearly equal chances. We shall assume, therefore, that on the average it remains unchanged. Hence we shall obtain by (7.1)

$$\bar{W}_s = \frac{1.79e^2}{\hbar c} \left[\log \frac{\delta_s}{\delta_0} + 0.715(\delta_0^{-2} - \delta_s^{-2}) - 2.83(\delta_0^{-3} - \delta_s^{-3}) \right]. \left. \begin{array}{l} \delta_s = 12.36, \\ \delta_0 = 1.81. \end{array} \right\} . \quad (7.6)$$

From (7.4), (7.6), and (5.3 a) the mass m_s for ${}_1\text{H}^3$ and ${}_2\text{He}^3$ will be easily obtained*.

We shall now consider the mass deficiency of the α -particle (${}_2\text{He}^4$).

It can be seen from the configuration of this nucleus, as shown in fig. 4, that neither of the two planes of symmetry, the plane of symmetry parallel to the central planes of the protons or that parallel to the central plane of the magneto-protons, has any preference for being a plane of a planetary orbit.

Therefore the stabilizing oscillations in a form of planetary motion seem to be unlikely. For this reason we shall assume symmetrical linear oscillations of the diametrically opposite particles.

Taking into account the observations at the end of section 5, we shall use the formula (5.9) for m_s without modifications.

In doing so we assume implicitly that only one pair of diametrical particles (say the protons) oscillate, while the other rests. This is, of course, unlikely. An exact analysis of such combined oscillations presents some difficulties, but a general examination of the problem suggests that in the case of such combined oscillations the potential energy will "migrate" from one pair of particles to the other, but that the average potential energy will remain the same as it would be if only one pair of particles oscillated while the other remained at rest. Therefore the results of our calculations of m_s would be still valid.

We shall now give the results of our calculations.

By subtracting the mass m_s of Table II. from the fundamental mass of Table I. we shall obtain the resultant

* See Table II. (p. 579).

mass. The result of this calculation is shown on Table III. alongside the results of Dr. Aston's measurements as quoted by Lord Rutherford in the course of his annual lectures at the Royal Institution in the Spring of 1936 *.

TABLE II.

	$\frac{1.0081 \Omega_s}{q_0+c}$	$\frac{1.0081 \bar{W}_0}{\hbar q_0+c}$	Mass deficiency, m_s .
${}_0\bar{n}^1$	0.0000165	-0.0000012	0.0000153
${}_0n^1$	0.0000350	-0.0000014	0.0000336
${}_1H^2$	0.00270	-0.000044	0.00265
	$-\frac{2 \times 1.0081}{\hbar q_0+c} K_s$	$\frac{1.0081 \bar{W}_s}{\hbar q_0+c}$	
${}_1H^3$	-0.0131	0.0214	0.0083
${}_2He^3$	-0.0131	0.0214	0.0083
${}_2He^4$	0.0298	—	0.0298

TABLE III.
Resultant Mass.

	Calculated.	Observed by Dr. Aston.
${}_1H^1$	—	1.0081
${}_0n^1$	1.00863	—
${}_0n^1$	1.00916	1.0091
${}_1H^2$	2.01464	2.0147
${}_1H^3$	3.0171	3.0171
${}_2He^3$	3.0171	3.0171
${}_2He^4$	4.0037	4.0039

This is a very satisfactory agreement.

* See also 'Nature,' cxxxvii. p. 396 (7th March, 1936).

8. *Conclusion.*

It can be seen from the above that the Whirl Theory of elementary particles, and the method of calculation of forces based on this theory, not only enables us to visualize a plausible structure of the neutron and the nuclei of the first two atomic numbers, but also leads to the numerical definition of their masses, which agrees remarkably with observations.

The theory also suggests the reasons why the mass deficiency is minimum when the number of positive units in the nucleus is a multiple of 4. It appears that in such nuclei the positive particles, according to their polarization, form two co-axial groups, with mutually perpendicular axes, in each of which there are two diametrically situated similar sub-groups. Every four particles which are symmetrically situated round the centre of the nucleus (point of intersection of both axes) form a system similar to the four particles in ${}_2\text{He}^4$. Consequently, for the same reasons as in the case of ${}_2\text{He}^4$, the particles have no planetary motion, but oscillate about their equilibrium position where the gradient of their holding forces is very high and, therefore, the potential energy "bound" in these oscillations is very large, and the larger, the greater is the total number of particles in the nucleus, *i. e.*, the higher is the atomic number. On the other hand, in the nuclei with the total number of particles not a multiple of 4, the exceeding 1, 2, or 3 particles are more loosely bound than the rest, which are arranged in an α -particle-like manner.

The close connexion of any four particles in the ${}_2\text{He}^4$ -like configuration, which are equi-distant, or nearly equi-distant, appears to be the cause of the formation of α -particles in radioactive disintegration.

It is probable that the analysis of the structure of heavier nuclei on the basis of the present theory will show that they contain, apart from neutrons, the neutron-like electro-polarized particles, *i. e.*, the combinations of the proton and the electrons in the sub-ground state. This would explain the ejection of electrons during the nuclear disintegration.

Of course, all these suggestions should be verified by a minute theoretical analysis and careful comparison with reliable observations.

Summary.

The structure of the neutron (n^1) and the simplest nuclei, namely deuteron (${}_1\text{H}^2$), the nuclei ${}_1\text{H}^3$ and ${}_2\text{He}^3$, and the α -particle (${}_2\text{He}^4$) was discussed on the basis of the author's theory of elementary particles (see Phil. Mag. xix. pp. 934-958, xx. pp. 417-468 and pp. 641-706 (1935)), according to which the elementary particles are certain axially symmetrical systems of Maxwell electromagnetic waves *in vacuo* which were called "electromagnetic whirls or simply 'Whirls'." The question of interaction between the so-called "contrapolarized" Whirls of equal size (like the electron and the positron, or magneto-electron) and between identical Whirls when they are situated co-axially, which was considered in the previous papers only in a general way, was discussed in detail. The transformation of energy and of mass, resulting from the interaction of elementary particles, was also discussed in greater detail. The holding forces and the kinetic and potential energies of the particles forming the nuclei were determined.

The difference between the sum of masses of elementary particles forming the neutron and the discussed nuclei and their resultant masses was calculated on the basis of the above-mentioned theory of elementary particles. The result proved to agree very well with the latest experimental data.

I am greatly indebted to Sir William Bragg, O.M., K.B.E., Pres.R.S., for his kind and encouraging interest in my work.

I wish to thank Miss A. C. Lewis, B.Sc., for reading the proofs.

I take this opportunity of thanking the Managers of the Royal Institution for giving me the facilities in the Davy-Faraday Laboratory for the experimental work with which the paper has a connexion.

Davy-Faraday Laboratory,
Royal Institution.
17th March, 1936.

XLVI. *Flexure of Beams of Polygonal Cross-section.*

By B. R. SETH, M.A., M.Sc., Ph.D.*

1. *Introduction.*

IN a recent paper † I have discussed the solution of a two-dimensional physical problem in which the boundary consists only of straight lines, and in which the solution can be represented by a function χ satisfying Laplace's equation and a boundary condition of the form

$$\left[\frac{\partial \chi}{\partial x} - (f_0 + f_1 + f_2 + \dots + f_n) \right] \cos(x\nu) + \left[\frac{\partial \chi}{\partial y} + (\phi_0 + \phi_1 + \phi_2 + \dots + \phi_n) \right] \cos(y\nu) = 0, \quad (1)$$

where f_n and ϕ_n are homogeneous integral polynomials of the n th degree in x and y , and ν denotes the direction of the normal drawn to the boundary.

If α_p is the argument of dz when the p th side of the boundary is described in the positive sense by the point whose affix is $z (=x+iy)$, we know that (1) can be put in the form

$$X_{n+1} \sin(n+1)\alpha_n + Y_{n+1} \cos(n+1)\alpha_p = [n \{ \sin \alpha_p f_n (\cos \alpha_p, \sin \alpha_p) + \cos \alpha_p \phi_n (\cos \alpha_p, \sin \alpha_p) \}], \\ (p=1, 2, 3, \dots, m), \quad (2)$$

m being the number of different sides present in the boundary and X_{n+1} , Y_{n+1} being given by

$$X_{n+1} + iY_{n+1} = \Omega_{n+1} = \frac{d^{n+1}(\chi + i\psi)}{d^{n+1}z}. \quad (3)$$

If the z -area is externally bounded by a polygon, the solution of the problem is given by the relations

$$\frac{dz}{d\zeta} = A \prod_{r=1}^m (\zeta - \xi_r)^{\alpha_r - 1}, \quad (4)$$

$$\frac{d\Omega_{n+1}}{d\zeta} = CR(\zeta) \prod_{r=1}^m (\zeta - \xi_r)^{-(n+1)\alpha_r}, \quad (5)$$

where $\xi_1, \xi_2, \dots, \xi_m$ are the points on the real axis of ζ in the ζ -plane that correspond to the angular points of

* Communicated by the Author.

† Phil. Mag. (7) xx. pp. 632-40 (1935).

the z -figure, $\pi a_1, \pi a_2, \dots, \pi a_m$ are the internal angles at these points, $R(\zeta)$ is a rational integral function of the $[(n+1)m-2(n+2)]$ th degree with real coefficients, and A, C are in general complex constants.

I propose to use the above results to get the flexure solution for beams whose cross sections are :

- (i.) A rectangle. The solution for this case is well known. We shall show how the above method gives the same result.
- (ii.) An isosceles right-angled triangle.
- (iii.) An isosceles triangle containing an angle of 120° .
- (iv.) An equilateral triangle.
- (v.) A right-angled triangle containing an angle of 60° .

We shall assume that the value of the torsion function ϕ is known for (iii.) and (v.)*.

2. Boundary Condition satisfied by the Flexure Function.

Let $(\bar{x}, \bar{y})$ be the coordinates of G , the centre of gravity of the cross-section. Then, for loads W_x and $W_x P_{xy}/I_y$ acting through G parallel to the axes of x and y respectively, we can write the three non-vanishing stress-components $\dagger$ of Saint Venant's classical solution as $\ddagger$

$$\begin{aligned} \widehat{xz} = & \mu\tau_1 \left(\frac{\partial \phi}{\partial x} - y \right) \\ & + \frac{\mu W_x}{EI_y} \left[\frac{\partial \chi}{\partial x} - \frac{1}{2}\sigma(x-\bar{x})^2 - (1-\frac{1}{2}\sigma)(y-\bar{y})^2 \right], \quad (6.1) \end{aligned}$$

$$\widehat{yz} = \mu\tau_1 \left(\frac{\partial \phi}{\partial y} + x \right) + \frac{\mu W_x}{EI_y} \left[\frac{\partial \chi}{\partial y} - (2+\sigma)(x-\bar{x})(y-\bar{y}) \right], \quad (6.2)$$

$$\widehat{zz} = -\frac{W_x}{I_y} (x-\bar{x})(l-z), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.3)$$

where in the usual notation $\chi(x, y)$ is the flexure function, I_y the moment of inertia of the cross-section about an axis through G parallel to the y -axis, P_{xy} the product of inertia about axis through G , E the Young's modulus,

* See Seth, *Quart. J. of Math.* v. pp. 161-171 (1934).

$\dagger$ These can be adapted from those given in Love's 'Treatise on Mathematical Theory of Elasticity' (1927 ed.), pp. 330-334.

$\ddagger$ The z -coordinate, which will be only used in (6), should not be confused with the complex variable z used before.

τ_1 a constant, l the length of the beam, and σ the Poisson's ratio.

The boundary condition

$$\widehat{xz} \cos(x\nu) + \widehat{yz} \cos(y\nu) = 0 \quad . \quad . \quad . \quad (7)$$

gives, after we have put

$$\chi = \chi_1 + (H_1 + \sigma \bar{y})\phi + x\left\{\frac{1}{2}\sigma \bar{x}^2 + (1 - \frac{1}{2}\sigma)\bar{y}^2\right\} + (2 + \sigma)\bar{x}\bar{y}y \\ - 2\bar{y}xy - \frac{1}{2}\sigma \bar{x}(x^2 - y^2) - \frac{1}{3}(1 - \frac{1}{2}\sigma)(x^3 - 3xy^2), \quad (8)$$

$$\left[\frac{\partial \chi_1}{\partial x} + H_1 y - x^2\right] \cos(x\nu) \\ + \left[\frac{\partial \chi_1}{\partial y} - H_1 x - 2\sigma xy + 2(1 + \sigma)\bar{x}y\right] \cos(y\nu) = 0, \quad . \quad (9)$$

which is of the same form as (1), the boundary condition satisfied by ϕ being

$$\left(\frac{\partial \phi}{\partial x} - y\right) \cos(x\nu) + \left(\frac{\partial \phi}{\partial y} + x\right) \cos(y\nu) = 0, \quad . \quad (10)$$

and H_1 being an undetermined *real* constant.

In like manner we have for loads $W_y P_{xy}/I_x$ and W_y acting through G parallel to the axes of x and y respectively

$$\chi = \chi_2 + (H_2 - \sigma \bar{x})\phi + \left\{(1 - \frac{1}{2}\sigma)\bar{x}^2 + \frac{1}{2}\sigma \bar{y}^2\right\}y + (2 + \sigma)x\bar{x}\bar{y} \\ - 2\bar{x}xy + \frac{1}{2}\sigma \bar{y}(x^2 - y^2) + \frac{1}{3}(1 - \frac{1}{2}\sigma)(3x^2y - y^3), \quad . \quad (11)$$

$$\left[\frac{\partial \chi_2}{\partial x} - 2\sigma xy + 2(1 + \sigma)x\bar{y} + H_2 y\right] \cos(x\nu) \\ + \left[\frac{\partial \chi_2}{\partial y} - y^2 - H_2 x\right] \cos(y\nu) = 0, \quad . \quad . \quad (12)$$

I_x being the moment of inertia of the cross-section about an axis through G parallel to the x -axis.

In the first case (2) gives

$$X_3 \sin 3\alpha_p + Y_3 \cos 3\alpha_p = (1 - 2\sigma) \cos \alpha_p \sin 2\alpha_p, \quad (13)$$

and in the second

$$X_3 \sin 3\alpha_p + Y_3 \cos 3\alpha_p = -(1 - 2\sigma) \sin \alpha_p \sin 2\alpha_p, \quad (14)$$

3. Zeros of $R(\zeta)$.

In general it is not possible to find in advance the zeros of $R(\zeta)$ for a given direction of load. But if the load acts along an axis of symmetry of the cross-section (if it

has any), some of the zeros of $R(\zeta)$ can be readily determined. For example, when the cross-section is triangular $R(\zeta)$ is linear, and can, therefore, be taken in the form $(\zeta - \xi_4')$, where ξ_4' is real. (5) now becomes

$$\frac{d\Omega_3}{d\zeta} = \frac{C(\zeta - \xi_4')}{(\zeta - \xi_1)^{3a_1}(\zeta - \xi_2)^{3a_2}(\zeta - \xi_3)^{3a_3}} \quad (15)$$

If the triangle is isosceles with the vertex at a_2 , then for a load acting along its axis of symmetry the zero ξ_4' must coincide with ξ_2 . (15) now takes the form

$$\frac{d\Omega_3}{d\zeta} = \frac{C}{(\zeta - \xi_1)^{3a_1}(\zeta - \xi_2)^{3a_2-1}(\zeta - \xi_3)^{3a_3}} \quad (16)$$

In like manner we can determine the zeros for a load acting along an axis of symmetry of a rectangular cross-section which we now proceed to discuss.

We shall require the values of $X_1 = \partial\chi_1/\partial x$, $Y_1 = -\partial\chi_1/\partial y$ at the angular points of the z -figure to determine the constants of integration. For $a > 1^*$, X_1 and Y_1 are both infinite at an angular point. For $a \leq 1^*$, X_1 and Y_1 are both finite, and since the boundary conditions (9) and (12) are to hold good for an infinite direction of ν at an angular point, we have for (13)

$$X_1 = x^2 - H_1y, \quad Y_1 = -2(1 + \sigma)\bar{x}y + 2\sigma xy + H_1x, \quad (17.1)$$

and for (14)

$$X_1 = -2(1 + \sigma)x\bar{y} + 2\sigma xy - H_2y, \quad Y_1 = y^2 + H_2x. \quad (17.2)$$

4. Rectangular Cross-section.

In this case if we put $\xi_1 = -k/k'$, $\xi_2 = 0$, $\xi_3 = k'/k$, $\xi_4 = \infty$, (4) becomes

$$z = A \int \left[\zeta \left(\zeta + \frac{k}{k'} \right) \left(\zeta - \frac{k'}{k} \right) \right]^{-\frac{1}{2}} d\zeta + B. \quad (18.1)$$

Adjusting the constants A and B, we get

$$\zeta = \frac{1}{kk'} ds^2(\tfrac{1}{2}z, k) \quad (18.2)$$

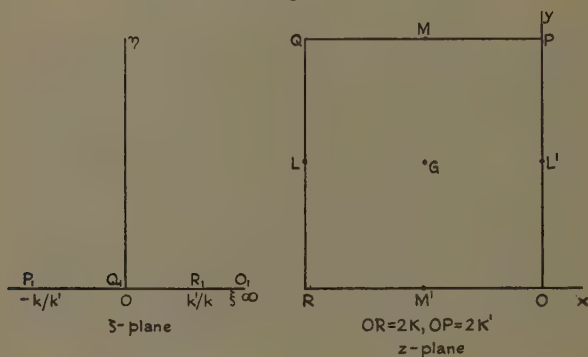
in Glaisher's notation, k' being the complementary modulus.

* This can be seen from the value of $\chi + i\psi$ near an angular point obtained in my paper in *Phil. Mag.* quoted before.

If $4K$, $4iK'$ be the complete periods, the coordinates of the angular points O , P , Q , R (see fig. 1) can be taken as o , $2iK'$, $-2K+2iK'$, and $-2K$ respectively.

Let us suppose that the load acts through G parallel to the axis of y . $R(\zeta)$ is now of the fourth degree. From considerations of symmetry we see that $R(\zeta)$ should vanish at the points L , G , and L' . G is given by $z = -K + iK'$, and substituting this value in (18.2) we get the corresponding value of ζ as i . Since the coefficients of all terms in $R(\zeta)$ are to be real, it must also vanish for

Fig. 1.



$\zeta = -i$. L , L' are given by $z = iK'$, $\zeta = -(1+k)/k'$ and $z = -2K + iK'$, $\zeta = (1-k)/k'$ respectively. Thus we have

$$k'^2 R(\zeta) = (\zeta^2 + 1)(k'\zeta + k + 1)(k'\zeta + k - 1). \quad (19)$$

With the help of (18.1), (18.2), and (19) we can write (5) as*

$$\frac{d\Omega_3}{dz} = C \frac{(\zeta^2 + 1)(k'\zeta + k + 1)(k'\zeta + k - 1)}{\zeta(k'\zeta + k)(k\zeta - k')}. \quad (20)$$

Now

$$cs^2 \frac{1}{2} z = k'(k\zeta - k'), \quad ns^2 \frac{1}{2} z = k(k'\zeta + k), \quad dn^2 \frac{1}{2} z = k'\zeta / (k'\zeta + k). \quad (21)$$

* Any constant factors which appear will be supposed to have been absorbed in C .

Substituting these values in (20) we get

$$\frac{d\Omega_3}{dz} = C \frac{(1 - 2k^2 \operatorname{sn}^2 \frac{1}{2}z + k^2 \operatorname{sn}^4 \frac{1}{2}z)(1 - k^2 \operatorname{sn}^4 \frac{1}{2}z)}{4 \operatorname{sn}^2 \frac{1}{2}z \operatorname{cn}^2 \frac{1}{2}z \operatorname{dn}^2 \frac{1}{2}z} = C \frac{\operatorname{dn} z}{\operatorname{sn}^2 z},$$

$$i. e., \quad \Omega_3 = C \cdot \operatorname{cs}(z) + D, \quad . \quad . \quad . \quad . \quad (22)$$

or, referred to L as the origin and using the Fourier's expansion for $\operatorname{dn} z$,

$$\begin{aligned} \Omega_3 &= iC \operatorname{dn} z + D \\ &= iC \sum_{n=1}^{\infty} \frac{\cos n\pi z/K}{\cosh n\pi K'/K} + D, \end{aligned}$$

which, if G is taken as the origin, becomes

$$\Omega_3 = iC \sum_{n=1}^{\infty} \frac{(-1)^n \cos n\pi z/K}{\cosh n\pi K'/K} + D. \quad . \quad . \quad (23)$$

Integrating (23) twice in succession, we get

$$\begin{aligned} \Omega_1 = X_1 + iY_1 &= -iC \sum_{n=1}^{\infty} \frac{K^2(-1)^n}{n^2\pi^2} \frac{\cos n\pi z/K}{\cosh n\pi K'/K} \\ &\quad + \frac{1}{2}Dz^2 + Ez + iF. \quad . \quad . \quad . \quad . \quad (24) \end{aligned}$$

Since the load acts along a coordinate axis of symmetry (G being now the origin), we have $\bar{x} = y = 0$. Also H_2 , the coefficient of ϕ in (11), must vanish. From (17.2) we have at an angular point

$$X_1 = 2\sigma xy, \quad Y_1 = y^2. \quad . \quad . \quad . \quad . \quad (25)$$

The four conditions at the angular point of the z -figure now give

$$D = -2i\sigma, \quad E = 0, \quad K'^2 = \frac{1}{6}CK'^2 + \sigma(K^2 - K'^2) - F, \quad (26)$$

where F can only be real.

Finally we have

$$\chi_2 + i\psi_2 = -iC \sum_{n=1}^{\infty} \frac{K^3(-1)^n}{n^3\pi^3} \frac{\sin n\pi z/K}{\cosh n\pi K'/K} - \frac{1}{3}i\sigma z^3 + iFz, \quad (27)$$

neglecting a constant of integration.

The condition that $\partial\chi_2/\partial y = K'^2$ on the sides $y = \pm K'$ gives

$$K'^2 = C \sum_{n=1}^{\infty} \frac{K^2(-1)^n}{n^2\pi^2} \cos \frac{n\pi x}{K} + \sigma(x^2 - K'^2) - F,$$

and, using the Fourier's expansion for $x^2(K > x > -K)$ and (26), we have

$$C = -4\sigma, \quad F = \frac{1}{3}\sigma K^2 - (1 + \sigma)K'^2.$$

The flexure function is now given from (27) and (11) as

$$\chi = -\frac{4\sigma K^3}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \cdot \frac{\cos(n\pi x/K) \sinh(n\pi y/K)}{\cosh n\pi K'/K} \\ + \frac{1}{6}(2 + \sigma)(3x^2y - y^3) - \left\{ \frac{1}{3}\sigma K^2 - (1 + \sigma)K'^2 \right\} y,$$

which is a well-known result.

5. Triangular Cross-section.

Since a triangular cross-section cannot have biaxial symmetry, the complete solution of the flexure problem requires the separate discussion for two different directions of the acting load. In the case of an isosceles triangle we can take these directions as the axis of symmetry and a straight line perpendicular to it. It can be shown that for a load acting perpendicular to the axis of symmetry of an isosceles triangle $\chi + i\psi$ has the form given by

$$\chi + i\psi A_1 z + A_2 z + A_3 z^3 + A_4(\phi + i\Psi), \quad \dots \quad (28)$$

where Ψ is the function conjugate to ϕ , and A_1, A_2, A_3, A_4 are all constants. This result does not hold good for an equilateral triangle unless $\sigma = \frac{1}{2}$. (28) is also found to be the form of the solution for any triangular cross-section and any direction of loading if $\sigma = \frac{1}{2}$ *.

We need, therefore, only discuss the flexure solution for a load acting along the axis of symmetry of (ii.) and (iii.). In an equilateral triangle we can take two medians as the two directions of loading; (v.) requires a separate treatment.

In all the four cases which we are going to discuss we shall put $\xi_1 = 1, \xi_2 = 0, \xi_3 = -1$.

6. Right-angled Isosceles Triangle.

Since the load is supposed to act along the axis of symmetry, we see from (16) that the Ω_3 -figure is also a right-angled triangle. Using (14), the areas that correspond to the upper half of the ζ -plane are shown shaded in fig. 2.

* These cases have been discussed in detail in a forthcoming paper in Proc. Lond. Math. Soc.

From (4) and (16) we have

$$\Omega_3 = Cz + D. \quad . \quad . \quad . \quad . \quad . \quad (29)$$

From fig. 2 we get

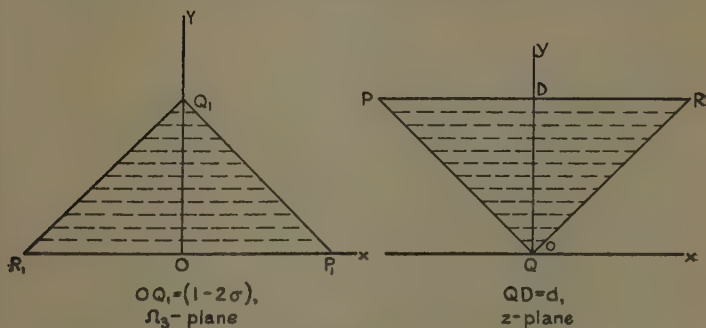
$$\Omega_3 = i(1-2\sigma) \text{ at } z=0, \quad \Omega_3 = \pm(1-2\sigma) \text{ at } z=d(\mp 1+i).$$

Thus $C = -(1-2\sigma/d), \quad D = i(1-2\sigma).$

Integrating (29) twice in succession, we have

$$X_1 + iY_1 = -\frac{1}{6d}(1-2\sigma)z^3 + \frac{1}{2}i(1-2\sigma)z^2 + Ez + F.$$

Fig. 2.



From (17.2)

$$X_1 = Y_1 = 0 \text{ at } z=0, \quad X_1 = \mp \frac{2}{3}(2-\sigma)d^2 - H_2d,$$

$$Y_1 = d^2 \pm H_2d \text{ at } z=d(\pm 1+i),$$

and hence

$$E = -\frac{2}{3}(1+\sigma)d, \quad F = H_2 = 0.$$

Finally we get

$$\chi_2 + i\psi_2 = -\frac{1}{24d}(1-2\sigma)z^4 + \frac{1}{6}i(1-2\sigma)z^3 - \frac{1}{3}d(1+\sigma)z^2,$$

which, combined with (11), gives *

$$\begin{aligned} \chi = -\frac{1}{24d}(1-2\sigma)(x^4 - 6x^2y^2 + y^4) + \frac{1}{6}(1+\sigma)(3x^2y - y^3) \\ - \frac{1}{3}d(x^2 - y^2) + \frac{2}{3}d^2\sigma y. \end{aligned}$$

* This case has been otherwise treated by me in Proc. Lond. Math. Soc. (2) xxxvii. pp. 502-511 (1934).

7. *Isosceles Triangle containing an Angle of 120°.*

In this case (4) becomes

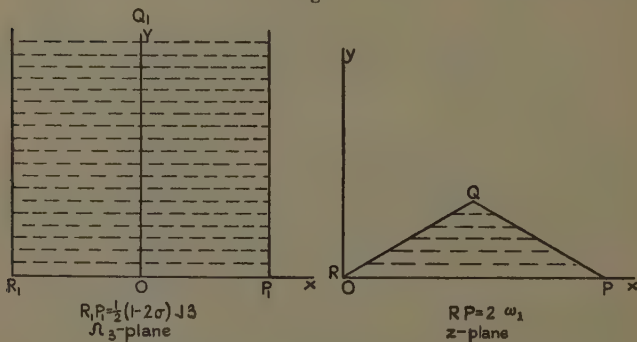
$$z = A \int \zeta^{-1/3} (\zeta^2 - 1)^{-5/6} d\zeta + B.$$

Adjusting the constants A, B, and writing* $\zeta = 1/f$, $h^3 = 1/(1-f^2)$, we have

$$\left. \begin{aligned} h &= \wp(z), & g_2 &= 0, & g_3 &= 4, \\ \zeta &= \frac{2\wp^{3/2}(z)}{\wp'(z)}, \end{aligned} \right\} \dots \dots (30)$$

and take the angular points P, Q, R (see fig. 3) as $2\omega_1$, $\omega_1(1+i/\sqrt{3})$, 0 respectively, ω_1 being the real half-period.

Fig. 3.



The load is again supposed to act along the axis of symmetry. From (16) we have

$$\frac{d\Omega_3}{d\zeta} = \frac{C}{\zeta(\zeta^2 - 1)^{1/2}}, \dots \dots (31)$$

and the corresponding areas are shown shaded in fig. 3.

Since $f = 1/\zeta$, (31) can be written as

$$\frac{d\Omega_3}{df} = - \frac{C}{(1-f^2)^{1/2}},$$

$$i. e., \quad \Omega_3 = -C \tan^{-1} \left\{ \frac{1}{2} \wp'(z) \right\} + D.$$

* Cf. Love, Amer. J. of Math. xi. p. 166 (1889).

Again putting the value of ζ from (30) in (31), we also have

$$\frac{d\Omega_3}{dz} = -\frac{3C}{\wp(z)} = -\frac{3}{2}iC\{\zeta(z+\lambda) - \zeta(z-\lambda) - 2\zeta(\lambda)\}, \quad (32)$$

where $\wp(\lambda)=0$, $\wp'(\lambda)=2i$, λ being the value of z when $\wp(z)=0$, i. e.,

$$\lambda = \omega_1(1+i/\sqrt{3}) = \frac{2}{3}(\omega_1 - \omega_3) = \omega_1 + \frac{1}{3}\omega_2',$$

ω_2' being the purely imaginary half-period.

To determine C we observe from the figure in the Ω_3 -plane that, as ζ passes through the value 0, Ω_3 increases by $-\frac{1}{2}(1-2\sigma)\sqrt{3}$. This increment can also be obtained from (31) by putting $\zeta = \epsilon e^{i\theta}$, and making $\epsilon \rightarrow 0$. Thus we get

$$C = \pm \frac{1}{2\pi}(1-2\sigma)\sqrt{3}.$$

Presently we shall show that it is the negative sign which we should take.

The Fourier's series for $\zeta(z)$ * is

$$\zeta(z) = \frac{\eta_1 z}{\omega_1} - \frac{\pi}{2\omega_1} \cot \frac{\pi z}{2\omega_1} = \frac{2\pi}{\omega_1} \sum_{r=1}^{\infty} \frac{q^{2r}}{1-q^{2r}} \sin \frac{r\pi z}{\omega_1},$$

$$\left\{ \left| q^2 \exp\left(\pm \frac{i\pi z}{\omega_1}\right) \right| < 1 \right\}, \quad . \quad . \quad . \quad (33)$$

where $q = \exp(i\pi\omega_2/\omega_1) = -i \exp(-\frac{1}{2}\pi\sqrt{3})$,

ω_2 being the half-period corresponding to e_2 , and given by

$$\omega_2 = -\frac{1}{2}\omega_1(1-i\sqrt{3}).$$

Since $z = x + iy$, the series in (33) is valid in the region

$$-\sqrt{3} < y/\omega_1 < \sqrt{3},$$

and in our figure $0 \leq y \leq \omega_1/\sqrt{3}$. Hence we can expand both $\zeta(z+\lambda)$ and $\zeta(z-\lambda)$ in Fourier series valid throughout the region occupied by the z -triangle. Substituting these expansions in (32), and integrating, we get

$$\Omega_3 = -\frac{3}{2}iC \left[\frac{2z}{\omega_1} \{ \eta_1 \lambda - \omega_1 \zeta(\lambda) \} + \log \sin \frac{\pi(z+\lambda)}{2\omega_1} \right. \\ \left. - \log \sin \frac{\pi(z-\lambda)}{2\omega_1} + 4 \sum_{r=1}^{\infty} \frac{q^{2r}}{r(1-q^{2r})} \sin \frac{r\pi\lambda}{\omega_1} \sin \frac{r\pi z}{\omega_1} \right] + D. \quad (34)$$

* See Tannery and Molk, *Fonctions elliptiques*, iii. p. 118 (1898).

With the help of the elliptic transformation of the third order we can establish the relations*

$$\zeta(\frac{2}{3}\omega_2') = \frac{2}{3}\eta_2' = -\frac{2}{3}i\eta_1\sqrt{3}, \quad . \quad . \quad . \quad (35.1)$$

$$\zeta(\omega_1 + \frac{1}{3}\omega_2') = \zeta(\lambda) = \eta_1 + \frac{1}{3}\eta_2' = \eta_1(1 - i/\sqrt{3}), \quad (35.2)$$

$$\eta_1\omega_1 = \frac{1}{3}\eta_2'\omega_2' = \frac{1}{6}\pi\sqrt{3}, \quad . \quad . \quad . \quad (35.3)$$

so that

$$\eta_1\lambda - \omega_1\zeta(\lambda) = \eta_1\omega_1(1 + i/\sqrt{3}) - \eta_1\omega_1(1 - i/\sqrt{3}) = \frac{1}{3}i\pi. \quad (36)$$

Again

$$\left. \begin{aligned} \log \frac{\sin \{ \pi(z + \lambda)/2\omega_1 \}}{\sin \{ \pi(z - \lambda)/2\omega_1 \}} &= i\pi \left(1 - \frac{z}{\omega_1} \right) \\ &+ \log \frac{1 - \exp \{ i\pi(z + \lambda)/\omega_1 \}}{1 - \exp \{ -i\pi(z - \lambda)/\omega_1 \}} \\ &= i\pi \left(1 - \frac{z}{\omega_1} \right) - 2i \sum_{r=1}^{\infty} \frac{q^{2/3r}}{r} \sin \frac{r\pi z}{\omega_1}. \end{aligned} \right\} \quad (37)$$

Since y is always positive in the region occupied by the z -triangle and is not greater than $\omega_1/\sqrt{3}$, we have

$$|\exp \{ i\pi(z + \lambda)/\omega_1 \}| < 1, \quad \text{and} \quad |\exp \{ -i\pi(z - \lambda)/\omega_1 \}| \leq 1.$$

Hence (37) is valid throughout the region in question in the sense that both of its sides become infinite for $z = \lambda$.

From the figure in the Ω_3 -plane we have $\Omega_3 = 0$ at $z = \omega_1$, $\Omega_3 = -\frac{1}{4}(1 - 2\sigma)\sqrt{3}$ at $z = 0$, $\Omega_3 = \frac{1}{4}(1 - 2\sigma)\sqrt{3}$ at $z = 2\omega_1$. Substituting these values in (34), and using (36) and (37), we get

$$C = -\frac{1}{2}(1 - 2\sigma)\sqrt{3}/\pi, \quad D = \frac{1}{2}(1 - 2\sigma)\sqrt{3}.$$

Integrating (34) twice in succession we have

$$\begin{aligned} X_1 + iY_1 &= \frac{3\sqrt{3}}{2\pi^3} \omega_1^2 (1 - 2\sigma) \sum_{r=1}^{\infty} \frac{q^{2/3r}(q^{2/3r} - 1)}{r^3(1 - q^{2r})} \sin \frac{r\pi z}{\omega_1} \\ &+ \frac{1}{8}(1 - 2\sigma)\sqrt{3} \left(\frac{1}{3} \frac{z}{\omega_1} - 1 \right) z^2 + Ez + F. \end{aligned} \quad (38)$$

From (17.2) we have $X_1 = Y_1 = 0$ at $z = 0$; $X_1 = -\frac{4}{9}(1 + \sigma)\omega_1^2\sqrt{3}$, $Y_1 = 2H_2\omega_1$ at $z = 2\omega_1$; $X_1 = -\frac{2}{9}(1 - 2\sigma)\omega_1^2\sqrt{3}$

* See Greenhill, 'Applications of Elliptic Functions,' pp. 315-316. He has taken $g_3 = 1$. In our case $g_3 = 4$, and hence his results have to be slightly modified.

$-\frac{1}{3}H_2\omega_1\sqrt{3}$, $Y_1=\frac{1}{3}\omega_1^2+H_2\omega_1$, at $z=\omega_1(1+i/\sqrt{3})$. These conditions give

$$F=0, \quad H_2=\sigma\omega_1, \quad E=-\omega_1\left\{\frac{1}{36}(5+14\sigma)\sqrt{3}+i\sigma\right\},$$

and the identity *

$$\frac{2\pi^3}{81}\sqrt{3}=\sum_{r=1}^{\infty}\frac{(1-q^{2/3r})(1-q^{4/3r})}{r^3(1-q^{2r})}. \quad \cdot \quad \cdot \quad (39)$$

Finally we have

$$\begin{aligned} \chi_2+i\psi_2 = & -\frac{3\sqrt{3}}{2\pi^4}\omega_1^3(1-2\sigma)\sum_{r=1}^{\infty}\frac{q^{2/3r}(q^{2/3r}-1)}{r^4(1-q^{2r})}\cos\frac{r\pi z}{\omega_1} \\ & +\frac{1}{24}(1-2\sigma)\sqrt{3}\left(\frac{1}{4}\frac{z}{\omega_1}-1\right)z^3 \\ & -\frac{1}{2}\omega_1\left\{\frac{1}{36}(5+14\sigma)\sqrt{3}+i\sigma\right\}z^2, \end{aligned}$$

which, combined with (11), gives χ .

8. Equilateral Triangle.

In this case (4) becomes

$$z=A\int\{\zeta(\zeta^2-1)\}^{-2/3}d\zeta+B.$$

Putting $\zeta=1/f$, $h^3=1-f^2$, and adjusting the constants, we write †

$$\left. \begin{aligned} h &= \wp(z), \quad g_2=0, \quad g_3=4, \\ \zeta &= \frac{2i}{\wp'(z)}, \end{aligned} \right\} \quad \cdot \quad \cdot \quad \cdot \quad (40)$$

and take the angular points P, Q, R as $2i\omega_1/\sqrt{3}$, 0, $\omega_1(1+i/\sqrt{3})$ respectively.

(a) For a load acting along LR (see fig. 4) we have from (16)

$$\frac{d\Omega_3}{d\zeta} = \frac{C}{\zeta(\zeta+1)}, \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (41)$$

and the corresponding areas are shown shaded in figs. 4 and 4a.

From (40) and (41) we have

$$\Omega_3 = -C \log [-\frac{1}{2}i\{\wp'(z)+2i\}] + D,$$

* This results from the fact that H_2 is real.

† Love, *loc. cit.* pp. 160-161.

which can also be written as

$$\frac{d\Omega_3}{dz} = -\frac{3}{2} C \frac{\wp'(z) - 2i}{\wp(z)} \quad \dots \quad (42)$$

By considering the increment in Ω_3 as ζ passes through the value 0 or -1 we get

$$C = \frac{3}{4}i(1-2\sigma)/\pi.$$

Now since

$$\frac{1}{2} \frac{\wp'(z) - 2i}{\wp(z)} = \zeta(z-\lambda) - \zeta(z) + \zeta(\lambda), \quad \dots \quad (43)$$

where $\wp(\lambda) = 0$, and $\wp'(\lambda) = -2i$, we have

$$\frac{d\Omega_3}{dz} = -\frac{9i}{4\pi} (1-2\sigma) \{ \zeta(z-\lambda) - \zeta(z) + \zeta(\lambda) \}, \quad \dots \quad (44)$$

Fig. 4 b.

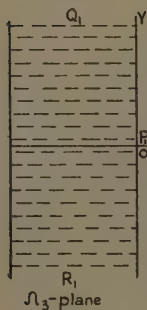


Fig. 4 a.

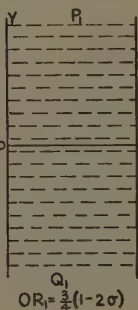
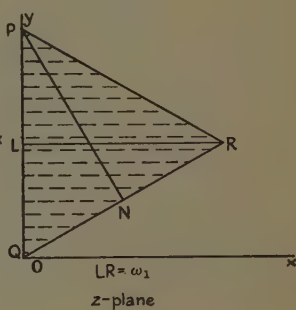


Fig. 4.



λ is the value of z when $\wp(z) = 0$. But $\wp(z) = 0$ at both the angular points P and R . To choose the proper value of λ we observe that Ω_3 is to be infinite for P and not for R (see fig. 4 a). Hence $\lambda = 2i\omega_1/\sqrt{3} = \frac{2}{3}\omega_2'$, ω_2' being the purely imaginary half-period as before.

The condition of convergency given with (33) holds good for both the expansions of $\zeta(z-\lambda)$ and $\zeta(z)$ in Fourier Series throughout the region occupied by the z -triangle. Substituting these expansions in (44), integrating, and using (36), we have

$$\Omega_3 = -\frac{9i}{4\pi} (1-2\sigma) \left[-\frac{2}{3} i\pi \frac{z}{\omega_1} + \log \sin \frac{\pi}{2\omega_1} (z-\lambda) - \log \sin \frac{\pi z}{2\omega_1} - 4 \sum_{r=1}^{\infty} \frac{q^{2r}}{r(1-q^{2r})} \sin \frac{r\pi\lambda}{2\omega_1} \sin \frac{r\pi}{\omega_1} (z-\frac{1}{2}\lambda) \right] + D. \quad (45)$$

Proceeding as in (37) we find

$$\log \frac{\sin \{\pi(z-\lambda)/2\omega_1\}}{\sin \{\pi z/2\omega_1\}} = \pi(\tfrac{1}{3}\sqrt{3}-i) + \frac{i\pi z}{\omega_1} + 2i \sum_{r=1}^{\infty} \frac{(-1)^r q^{2/3r}}{r} \sin \frac{r\pi}{\omega_1} (z - \tfrac{1}{2}\lambda). \quad (46)$$

From fig. 4a we have $\Omega_3=0$ at $z=\tfrac{1}{2}\lambda$, $\Omega_3=\tfrac{3}{4}(1-2\sigma)$ at $z=\omega_1+\tfrac{1}{2}\lambda$. Substituting these values in (45), and using (46), we get

$$D = \tfrac{1}{4}(1-2\sigma)(9+2i\sqrt{3}).$$

Integrating (45) twice in succession we have

$$X_1 + iY_1 = -\frac{9}{2\pi^3}\omega_1^2(1-2\sigma) \sum_{r=1}^{\infty} \left[\frac{(-1)^r q^{2/3r}(1-q^{2/3r})}{r^3(1-q^{2r})} \sin \frac{r\pi}{\omega_1} (z - \tfrac{1}{2}\lambda) \right] + \tfrac{1}{8}(1-2\sigma) \left(\frac{z}{\omega_1} - i\sqrt{3} \right) z^2 + Ez + F.$$

From (17.1) the three conditions at the three angular points give $F = -\tfrac{1}{8}i\omega_1^2\sqrt{3}(1-2\sigma)$, $H_1 = -\tfrac{2}{3}\sigma\omega_1\sqrt{3}$, $E = \tfrac{1}{4}\omega_1(3+2\sigma) - \tfrac{1}{3}i\sigma\omega_1\sqrt{3}$, and the identity which we have obtained in (39).

Finally we have

$$\chi_1 + i\psi_1 = \frac{9}{2\pi^4}\omega_1^3(1-2\sigma) \sum_{r=1}^{\infty} \frac{(-1)^r q^{2/3r}(1-q^{2/3r})}{r^4(1-q^{2r})} \cos \frac{r\pi}{\omega_1} (z - \tfrac{1}{2}\lambda) + \frac{1}{96}(1-2\sigma) \left(\frac{3z}{\omega_1} - 4\sqrt{3} \right) z^3 + \frac{1}{24} \{ 3(3+2\sigma) - 4i\sigma\sqrt{3} \} \omega_1 z^2,$$

which, combined with (8), gives the value of χ .

(b) For a load acting along PN (see fig. 4) (16) becomes

$$\frac{d\Omega_3}{d\zeta} = \frac{C}{\zeta(\zeta-1)},$$

$$i. e., \quad \Omega_3 = C \log [\tfrac{1}{2}i\{\wp'(z) - 2i\}] + D,$$

$$\text{or} \quad \frac{d\Omega_3}{dz} = \frac{3}{2} C \frac{\wp'(z) + 2i}{\wp(z)}.$$

Combining (13) and (14) our boundary condition now becomes

$$X_3 \sin 3\alpha_p + Y_3 \cos 3\alpha_p = -\tfrac{1}{2}(\sqrt{3}+1)(1-2\sigma) \sin 2\alpha_p (\cos \alpha_p + \sqrt{3} \sin \alpha_p),$$

and the corresponding areas are shown shaded in figs. 4 and 4b.

Proceeding as in (a) we find

$$\frac{d\Omega_3}{dz} = \frac{9i}{4\pi} (1-2\sigma)(\sqrt{3}+1)[\zeta(z-\lambda)-\zeta(z)+\zeta(\lambda)],$$

where now $\wp(\lambda)=0$, $\wp'(\lambda)=2i$, and $\lambda=\omega_1+\frac{1}{3}\omega_2$.

The rest of the analysis follows the same lines as in (a).

9. Right-angled Triangle containing an Angle of 60° .

In this case we obtain from (4)

$$z=A\int \zeta^{-1/2}(\zeta-1)^{-5/6}(\zeta+1)^{-2/3}d\zeta+B_1.$$

Putting $\zeta=1/f$, $h^3(f+1)=f-1$, and adjusting the constants, we write*

$$\left. \begin{aligned} h &= \wp(z), & g_2 &= 0, & g_3 &= 4, \\ \zeta &= \frac{1-\wp^3(z)}{1+\wp^3(z)}, \end{aligned} \right\} \quad \dots \quad (47)$$

and take the angular points P, Q, R (see fig. 5) as 0, ω_1 , $\omega_1(1+i/\sqrt{3})$ respectively.

From (15) we have

$$\frac{d\Omega_3}{d\zeta} = \frac{C(\zeta-\xi_4')}{(\zeta-1)\zeta^{1/2}(\zeta+1)^{1/2}} \quad \dots \quad (48)$$

ξ_4' is different for different directions of loading. For some direction it will vanish. Let us suppose this direction is obtained by combining the solutions for loads $W_x+W_yP_{xy}/I_x$ and $W_y+W_xP_{xy}/I_y$ acting along the axis of x and y respectively. Our boundary condition now becomes

$$\begin{aligned} X_3 \sin 3\alpha_p + Y_3 \cos 3\alpha_p \\ = (1-2\sigma) \sin 2\alpha_p (A_1 \cos \alpha_p - A_2 \sin \alpha_p), \end{aligned}$$

where $A_1=\mu W_x/EI_y$, $A_2=\mu W_y/EI_x$, and the corresponding areas are shown shaded in figs. 5, 5a, 5b.

Putting the value of ζ from (47) in (48), we get

$$\begin{aligned} \frac{d\Omega_3}{dz} &= \frac{3Ci\sqrt{2}}{\wp(z)} \\ &= -\frac{3}{2}C\sqrt{2}[\zeta(z+\lambda)-\zeta(z-\lambda)-2\zeta(\lambda)], \quad \dots \quad (49) \end{aligned}$$

$$i. e., \quad \Omega_3 = -\frac{1}{2}C\sqrt{2} \cosh^{-1}\{1-2/\wp^3(z)\} + D.$$

Here $\wp(\lambda)=0$, $\wp'(\lambda)=2i$, and $\lambda=\omega_1(1+i/\sqrt{3})$.

* Love, *loc. cit.* p. 164.

Proceeding as in Section 7, we find

$$C = -\frac{1}{4}\sqrt{6} \frac{i}{\pi} (1-2\sigma)(A_2 - A_1\sqrt{3}),$$

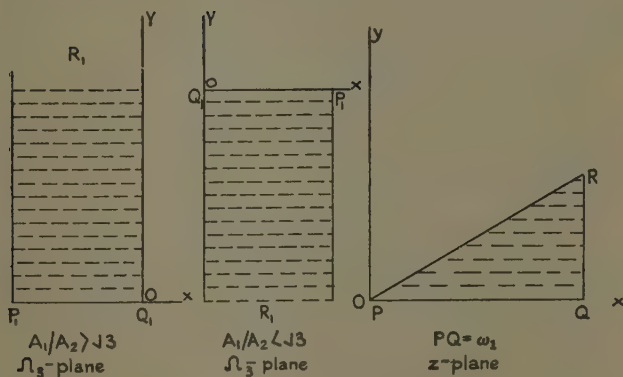
$$\Omega_3 = \frac{3\sqrt{3}}{2\pi} (1-2\sigma)(A_2 - A_1\sqrt{3}) \sum_{r=1}^{\infty} \frac{q^{2/3r}(q^{2/3r}-1)}{r(1-q^{2r})} \sin \frac{r\pi z}{\omega_1}$$

$$+ \frac{1}{4}\sqrt{3}(1-2\sigma)(A_2 - A_1\sqrt{3}) \left(\frac{z}{\omega_1} - 1 \right),$$

Fig. 5 b.

Fig. 5 a.

Fig. 5.



$$X_1 + iY_1 = \frac{3\sqrt{3}}{2\pi^3} (1-2\sigma)(A_2 - A_1\sqrt{3}) \omega_1^2 \sum_{r=1}^{\infty} \left[\frac{q^{2/3r}(q^{2/3r}-1)}{r^3(1-q^{2r})} \right]$$

$$\sin \frac{r\pi z}{\omega_1} \Big] + \frac{1}{8} \sqrt{3}(1-2\sigma)(A_2 - A_1\sqrt{3}) \left(\frac{1}{3} \frac{z}{\omega_1} - 1 \right) z^2$$

$$+ \left[\frac{1}{4} A_1(3+2\sigma)\omega_1 - A_2 \left\{ i\sigma\omega_1 + \frac{1}{36} (5+14\sigma)\omega_1\sqrt{3} \right\} \right] z,$$

$$H_1 = 0, \quad A_2 H_2 = \sigma\omega_1,$$

and if

$$\chi_3 = A_1 \chi_1 + A_2 \chi_2,$$

$$\chi_3 + i\psi_3 = \frac{3\sqrt{3}}{2\pi^4} (1-2\sigma)(A_2 - A_1\sqrt{3}) \omega_1^3 \sum_{r=1}^{\infty} \left[\frac{q^{2/3r}(q^{2/3r}-1)}{r^4(1-q^{2r})} \right] \times$$

$$\begin{aligned} & \cos \frac{r\pi z}{\omega_1} \Big] + \frac{1}{24} \sqrt{3}(1-2\sigma)(A_2 - A_1\sqrt{3}) \left(\frac{1}{4} \frac{z}{\omega_1} - 1 \right) z^3 \\ & + \frac{1}{2} \omega_1 \left[\frac{1}{4} A_1(3+2\sigma) \right. \\ & \left. - A_2 \left\{ i\sigma + \frac{1}{36} (5+14\sigma) \omega_1 \sqrt{3} \right\} \right] z^2. \quad (50) \end{aligned}$$

The boundary condition satisfied by χ_3 can be obtained by multiplying (9) by A_1 , (12) by A_2 , and then adding the results. If we substitute the value of χ_3 from (50) in it, we find that it is satisfied over all the sides of the triangle for all values of A_1 and A_2 . Thus (50) constitutes the general solution for this cross-section. If θ is the angle the resultant load makes with the x -axis we have

$$\tan \theta = \frac{A_2 I_x + A_1 P_{xy}}{A_1 I_y + A_2 P_{xy}}.$$

In particular, if $A_1/A_2 = 1/\sqrt{3} = -\bar{y}/\bar{x}$, $\theta = \frac{1}{6}\pi$, and we get a simple result of the form

$$\chi_3 = C_1(x^2 - y^2) + C_2xy,$$

C_1 and C_2 being constants. It can be verified independently that this form of χ_3 holds good for a load acting through C_1 parallel to the hypotenuse of any right-angled triangle.

It is also interesting to note that the value of χ reduces to the form given in (28) in all the cases discussed in Sections 6, 7, 8, 9 if $\sigma = \frac{1}{2}$.

10. Summary.

To sum up, we have in the case of (ii.) for any direction of load

$$\chi + i\psi = A_1 z + A_2 z^2 + A_3 z^3 + A_4 z^4 + H(\phi + i\Psi).$$

For (iii.) and (v.) we have the general result

$$\chi + i\psi = A_1 z + A_2 z^2 + A_3 z^3 + A_4 z^4 + C \iiint \frac{dz^4}{\wp(z)} + H(\phi + i\Psi).$$

For (iv.) we should add another term of the form

$$\iiint \log \wp(z) dz^3$$

to the above result.

XLVII. *The Use of High-Frequency Discharge-tubes as Electrical Counters.* By SAMUEL C. CURRAN, M.A., B.Sc., Metcalfe Research Fellow, Glasgow University *.

Introduction.

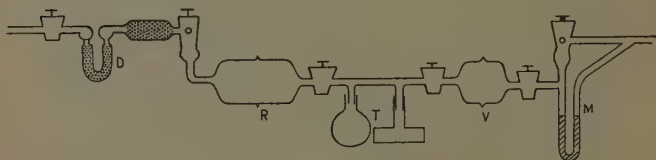
THE subject of the electrical counter is one which has received considerable attention since it was described in its first form by Rutherford and Geiger ⁽¹⁾, and various modifications and improvements have been made by numerous investigators from time to time. The principle embodied in the different types is the same—when the potential applied to a discharge-tube is less than that required for sparking the entry of a single ionizing particle will cause, under certain conditions, the momentary passage of a detectable current. In the prototype of the electrical counter, as described by Rutherford and Geiger ⁽¹⁾, one electrode was the wall of the cylindrical vessel and the other was a fine axial wire. This was changed to a hemispherical vessel with a central small ball by Geiger and Rutherford ⁽²⁾ later, and to a cylinder and sharp point by Geiger ⁽³⁾. The sharp point in the last case was specially sensitized before use. In nearly all the earlier types of counter what takes place is simply magnification of the ionization currents by collision in the strong electric field between the electrodes ⁽⁴⁾, and no “trigger action,” in which the energy stored in the capacity of the counting vessel is released by the entry of the particle, takes place. This means in these cases that the currents to be detected are extremely small, and considerable amplification has to be effected. Even in the point-cylinder type of counter working on the principle of “trigger action” the currents passing through the tube are still of the order of a few microamperes at the most, and amplification is required to make them audible in a loud-speaker for instance. The small area over which the rays can enter and be effectively counted is for many purposes a disadvantage of the point-cylinder type, and recently some experimenters have reverted to the wire type ⁽⁵⁾, which can be used for X-rays and γ -rays as well as α -particles and β -particles, the rays entering through the

* Communicated by Prof. E. Taylor Jones, D.Sc.

sides of the tube. In addition, it has been found possible to use plane parallel electrode tubes as counters ⁽⁶⁾, a steady potential being maintained across the electrodes. These tubes are erratic in their behaviour and have not the required stability of action.

In the following pages an account is given of experiments carried out on high-frequency discharge-tubes of various kinds, in which it has been found that such tubes can be adapted so that they act very efficiently as electrical counters of good sensitivity, and in which the "energy trigger ratio" is so large that a small loud-speaker may be included in the oscillator circuit to detect the counts as audible clicks. The use of the H.F. potentials in place of the D.C. potentials makes it possible to work the counter from a D.C. supply of 400 volts or even less, and thus the use of high steady potentials required

Fig. 1.



H.F. discharge-tube system.

in most other counters is avoided. The erratic behaviour associated with some D.C. discharge-tubes is absent for reasons given later, and the effective counting area for these tubes is very large.

Apparatus.

The pumping system consisted of a rotary oil-pump backing an oil-diffusion pump, using Apiezon oil B, and capable of producing a vacuum of the order of 10^{-5} mm. of mercury. It was attached to a manometer and various tubes shown in diagrammatic form in fig. 1. The gases admitted to the tubes were passed through a drying system of calcium chloride and phosphorus pentoxide.

In fig. 1 D represents the drying tubes, the gas being passed into the system at the left-hand side of the figure. R is a large reservoir for storing the dried gas and for

use in adjusting the gas pressure in the discharge-tubes T and V, these being separated to allow for pressure variation in the manner described by Thomson ⁽⁷⁾. The manometer M contains Apiezon oil B of very low vapour-pressure. Suppose V_A is the volume of V and V_B is the volume of T, and the manometer, when pumped out completely on the right-hand side, records a pressure p_0 cm. of mercury, as read by a travelling microscope. T is cut off from V, V is pumped out, and the gas in T is then shared between the vessels T and V. If the resulting pressure is p_1 cm. of mercury, then

$$p_1(V_A + V_B) = p_0 V_B,$$

$$i. e., \quad p_1 = p_0 \frac{V_B}{V_A + V_B} = k p_0 \text{ say.}$$

Thus k is determined from readings p_1 and p_0 , and proceeding with the same process, subsequent pressures,

$$p_2 = k p_1 = k^2 p_0, \quad p_3 = k^3 p_0, \quad \dots, \quad p_n = k^n p_0,$$

can all be calculated. The highest pressure which could be measured directly on M was 22 cm. of oil, or 1.41 cm. Hg, but by starting at higher pressures and taking the readings p_m and p_{m+1} when they were below this value pressures ranging from 7 cm. to .001 mm. of mercury were recorded.

The oscillator used for the generation of the H.F. potentials was of the push-pull type shown in fig. 2.

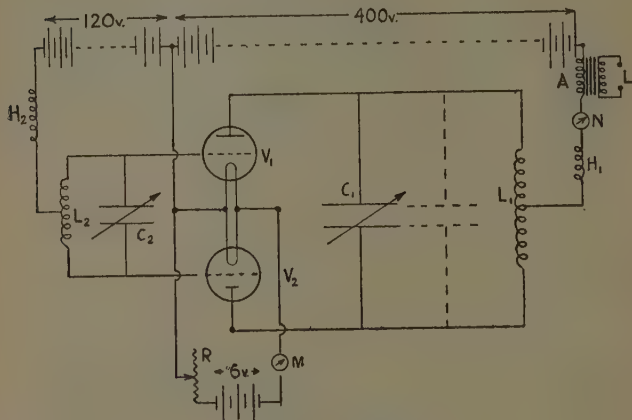
The matched valves V_1 and V_2 were Marconi-Osram L.S.6A, dissipating about 25 watts each. The oscillatory circuit $L_1 C_1$ was tuned to the grid circuit $L_2 C_2$ by variation of the capacity C_1 or C_2 according as circumstances required. H_1 and H_2 are two high-frequency chokes, A is an audio-frequency transformer, L is the loud-speaker, M is an ammeter measuring filament current, N is a milliammeter measuring plate current, and R is a pair of variable resistances giving fine adjustment of the output of the oscillator.

Three of the discharge-tubes which were found to act as electrical counters will be described, though several others were used in an attempt to find those most suitable for this purpose.

Fig. 3(a) is a sketch of tube I. This consisted of a large glass vessel G consisting of two parts sealed together

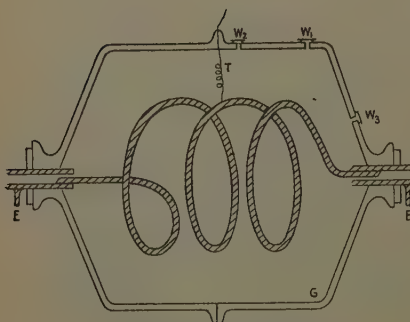
along ground flanges. Copper tubes bearing terminals E passed through the ends, and were connected by a coil of copper wire consisting of three turns with a fine

Fig. 2.



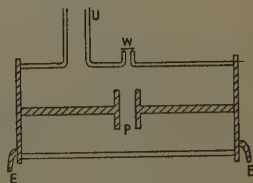
H.F. push-pull oscillator.

Fig. 3 (a).



Discharge-tube number I.

Fig. 3 (b).



Discharge-tube number II.

centre-tapping wire T connected to it, and passing through the side of the vessel at the joint. Three small windows through the glass wall, W_1 , W_2 , and W_3 , were covered by thin sheets of mica. The vessel G was 13 cm. in

diameter and 15 cm. long. The total length of the coil was 80 cm., and the wire had a cross-section of diameter 2.5 mm. Fig. 3 (b) shows tube II., which consisted of a glass vessel 20 cm. long, closed at its ends by two metal plates bearing terminals E and two rods which supported two copper plates P of 2 cm. diameter at a distance apart of 3 mm. The vessel was provided with a window W covered with mica and a pumping tube U. Tube III. was exactly similar to tube I., except that in place of a coil and centre-tapping T there were two large copper plates supported in the centre of the vessel, at a distance of 1 mm. from each other. Their diameters were 7.8 cm. each. It is to be noticed that in using tube I. the coil L_1 was replaced by the coil of the discharge-tube, the centre-tapping T going through H_1 to +400 volts. With this voltage across the valves it was possible to generate a peak potential of 1000 volts across the ends of the coil. When tube II. was used it was connected across L_1 as indicated by dotted lines in fig. 2. L_1 in this case was a coil of 20 turns, each turn 2.5 cm. in diameter, and of total length 8 cm. L_2 was usually a coil of 12 turns, each turn 5.6 cm. in diameter, the total length being 8.5 cm., while C_1 and C_2 were 100 $\mu\mu\text{F.}$ variable condensers. When tube III. was used the condenser C_1 was removed and the discharge-tube connected across L_1 . The capacity between the plates was about 40 $\mu\mu\text{F.}$, and tuning was carried out by means of C_2 . It was possible to have a peak potential of 1000 volts across tube III. in this case.

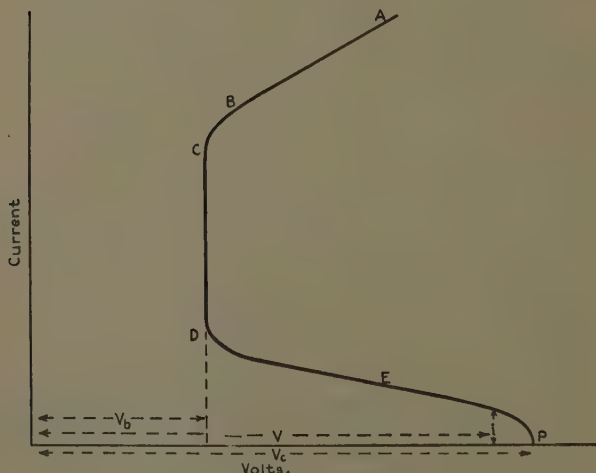
Measurements of voltage across the tubes were seldom required in absolute values, so a method giving relative values quickly and accurately was usually adopted. A loop of thick copper wire about 15 cm. in diameter was connected to the heater terminals of a vacuum thermocouple, which was in turn connected to a microammeter. It can be proved that the heater current set up in the loop when it is placed near the oscillator is proportional to the voltage generated, and therefore from a calibration curve connecting heater current and the current in microamperes in the thermocouple circuit it is possible to find relative values of the voltages from readings of the microammeter. This method has the additional advantage that the oscillator is unaffected in taking readings. In the case of the highest frequency employed,

$2.3 \cdot 10^7$ cycles per second, the use of an electrostatic voltmeter is not feasible because of the capacity it introduces into the system, but in several cases readings of the sparking potentials etc. were taken with the electrostatic voltmeter when working at frequencies of the order of 10^7 c./s. These were taken by connecting an electrostatic voltmeter, in series with a diode valve, across the electrodes of the tubes.

Theoretical Discussion.

Several investigators have studied the mode of action of various forms of electrical counters, in the elucidation

Fig. 4.



General volts-amperes characteristic.

of which many problems arise. The mode of action of the H.F. tubes can be best explained by comparison with that of the standard point-cylinder type. Appleton, Emeléus, and Barnett ⁽⁸⁾ carried out a series of experiments on this subject, and suggested the idea of threshold current. This idea was experimentally verified by Taylor ⁽⁹⁾, who carried their explanation a step further.

Thus suppose ABCDEP is the general volts-amperes characteristic of a discharge-tube, where AB is the statical boundary condition, CD is the region of normal cathode fall, and DP is the corona characteristic. Taylor showed that DP, the corona characteristic, is the same as the region of threshold current proposed by Appleton, Emeléus, and Barnett in their explanation of the action of the counter, and his explanation of the counting action is that "if in a discharge-tube having a voltage V across the electrodes, a threshold current i is produced either by external or internal ionizing factors, a self-sustained discharge will be initiated if, and only if the voltage V across the electrodes is equal to or greater than the voltage on the corona characteristic corresponding to the current i " ⁽¹⁰⁾. The threshold current is, of course, the current set up by magnification of ionization by collision in the electric field between the electrodes. This explains how the point-cylinder vessel works, since in this case the corona characteristic DEP lies very close to the axis near P before it finally cuts it there. Thus a small threshold current can have a considerable effect, allowing a discharge to be initiated at the potential V instead of V_c . With plane parallel electrode tubes the corona characteristic rises rather perpendicularly to the axis at P, and Taylor considers the threshold currents are insufficient to allow the discharge to pass at some potential V less than V_c . The action in this case is therefore considered to depend on polarization of the electrodes rather than on the shape of the volts-amperes characteristic near P, and since such polarization layers are erratic in their behaviour these counters are not very stable in their action. Electron liberation at the cathode causes neutralization of the charged layer locally and consequent lowering of the sparking potential to a value at which a flash may occur with the existing potential across the tube. Suppose now we consider a discharge-tube such as that of fig. 3 (a) or fig. 3 (b) with a high-frequency potential across the electrodes E. It appears that when such tubes act as counters either of the two actions described above must take place. Polarization effects in such H.F. tubes are very slight compared with those in D.C. tubes ⁽¹¹⁾, and it would seem to be necessary to explain the action of such tubes on the basis of corona currents. The corona régime has not been systematically

investigated for the H.F. discharge, but Thomson ⁽⁷⁾ has produced evidence of its existence. Now it is probable that here, as in D.C. parallel plate discharge-tubes, the corona characteristic rises steeply from the voltage axis at P (fig. 4), and the question is whether the threshold currents set up by collision on the entry of an ionizing particle are sufficiently large to bring about what is virtually a lowering of the sparking potential of the H.F. discharge-tube; for a tube such as that of fig. 3 (b), with a steady potential across the electrodes such an action is not possible, but the threshold currents brought about by ionization by collision when the steady potential is changed to a high-frequency potential might conceivably be of much greater magnitude, since in this case the ions oscillate to and fro under the influence of the alternating electric field. The opportunities for collisions in a very short interval of time in a field of frequency of the order of 10^7 c./s. are increased in this way, provided that the conditions are such that the loss of ions by contact with the electrodes etc. during the interval is small. It is possible that these conditions can be fulfilled, and the experiments on the H.F. tubes supported this conclusion. In this connexion mention may be made of an experiment of Sir J. J. Thomson ⁽¹²⁾, who showed that it is possible to light an electrodeless discharge by passing ultra-violet light through the discharge-tube when it is critically adjusted for sparking. This lends additional support to the hypothesis that some such action as that described above is responsible for bringing about a discharge on the entry of an ionizing particle into tube I. Since then it is unusual to have polarization effects greater than 1 or 2 volts in an H.F. discharge-tube, it seems reasonable to conclude that the initial action of an ionizing particle on entry is to cause a building up of the threshold current to a value equal to that of some part of the corona current characteristic, and that this process results in a glow discharge. Such an explanation does not preclude, however, the possibility that a very slight polarization of the electrodes, or an alteration in the field distribution of the electric force due to the existence of corona currents, plays some part in the mechanism of the flashing also. Indeed, it is quite possible that the action is a combination of both effects, just as Taylor concludes that in the case of the point-cylinder counter although the corona

can explain the action it does not preclude the chance of both processes being involved to a greater or lesser degree.

This brings us to the second important point in the working of any counter—the self-restoring property. The action in the case of the point-cylinder or the D.C. plane parallel electrode tube depends in an intimate way on the presence of large resistance in the circuit ⁽⁹⁾. The self-restoring property of the H.F. discharge-tube must be altogether different therefore, and is as follows. When a discharge passes through the tube its capacity is altered, and this has a decided effect on the tuning of the oscillator. The effect is equivalent to adding capacity to the anode oscillatory circuit. Thus when the counter is working the passage of a current is the same as the sudden addition of capacity to the oscillatory circuit with consequent detuning. If this is sufficient to lower the oscillatory voltage across the ends of the tube to a momentary value below that of the maintenance potential of the tube, then the discharge ceases to pass and a flash takes place. This was found to occur in practice, being especially marked at the lower pressures, and when it did not occur it was usually possible to obtain a flashing state of the discharge either by increasing the capacity C_1 or by decreasing the capacity C_2 (see fig. 2). Thus, to take an example, in one case the oscillator when tuned to resonance had a frequency of $1.11 \cdot 10^7$ c./s., and the frequency when it was detuned to act as a counter giving flashes was $1.07 \cdot 10^7$ c./s.

Experimental Procedure and Results.

Before using any of the tubes they were “conditioned.” The vessel was pumped out to as low a pressure as the pumps were capable of producing. It was then allowed to stand for some hours, and the pumping was resumed. This was repeated at intervals for a few days. The gas to be used was next passed through the drying system into the thoroughly evacuated discharge-tube, and the pressure was adjusted so that when the glow discharge was running it covered the plates in the case of tube II. or tube III. In the case of tube I. the discharge was run also. After passing the discharge for a few hours the tube was again exhausted, fresh gas passed in, and the whole process repeated. This was done

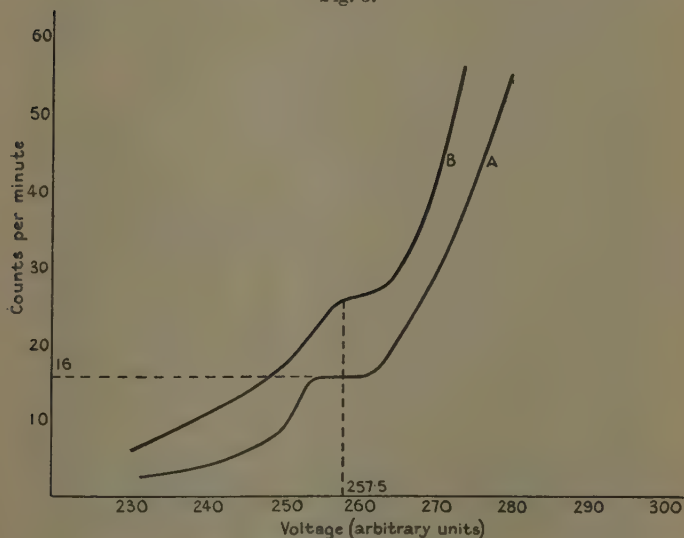
several times before the tubes were used as counters. In the case of the parallel plate tubes, tubes II. and III., this "conditioning" was found to have a material effect in improving the action. The plates were carefully cleaned and heated for some time in a Bunsen flame before assembly in the glass vessels.

Experiments carried out with tube I., which became L_1 of fig. 2, showed that with the 3-turn loosely wound coil the gas pressures at which discharges could be obtained were low. The range of pressure investigated was from $\cdot 27$ mm. Hg to $\cdot 001$ mm. Hg. (Experiments with other coils, for instance one of 20 turns about 10 cm. long, showed that although higher gas pressures could be used they did not give a good counting state.) A curve of sparking potential against pressure was determined for this range, and was of the usual form, showing the characteristic straight portion falling to either a single minimum⁽⁷⁾ or in some cases to two minima. It was found that the tube acted as a counter on either side of the critical pressure, $\cdot 02$ mm. Hg with air as the filling gas. With the oscillator adjusted so that the frequency was $2 \cdot 0 \cdot 10^7$ c./s., and with the pressure of the gas at $\cdot 05$ mm. Hg, a characteristic curve of counts per minute against voltage across the ends of the coil was plotted, as shown in fig. 5. The voltage was determined in relative values only by the thermocouple and galvanometer method described above. The voltage was raised slowly by adjustment of the filament resistance R of fig. 2, and at a certain value counts began to be heard. The discharges giving rise to these could scarcely be seen in a darkened room. With air they took the form of a dull blue flash confined almost entirely within the coil at the pressure $\cdot 05$ mm. Hg, in the neighbourhood of which tube I. was normally used. As the voltage was still further increased the number of counts per minute slowly increased, and finally the discharge became intermittent, flashing so rapidly that it produced a howl in the loud-speaker. This behaviour is exactly similar to that of the point-cylinder type. It is seen from fig. 5 that there is a part of the curve approximately parallel to the voltage axis, the counts per minute remaining steady through the range of voltage corresponding to 255 to 260. Thus the correct operating voltage of the tube is about the mid-point of this range, and is seen from fig. 5,

curve A, to be at 257.5. From this part of curve A we can determine the natural count for tube I., this being 16 per minute.

Curve A was determined when there were no radioactive substances present, but curve B was plotted with a weak source placed at a distance from the counter. As is to be expected curve B lies above curve A, the flattened portion being not so well defined. It is seen from the two curves that the source has brought about

Fig. 5.



Characteristic curve for tube I.

10 additional counts per minute. Tube I. was tested for radiation by bringing a radon capillary containing 1 millicurie into its vicinity. As it was brought nearer the rate of counting rose very rapidly and attained a high value. If we define the sensitivity of the counter to be measured by the distance at which 1 millicurie doubles the natural count, then for tube I. it is 3.5 metres. The counter was used to detect α -rays also, these passing into the vessel through the mica windows.

Experiments on tube II. were carried out on the same lines. It was found that with argon as the filling gas

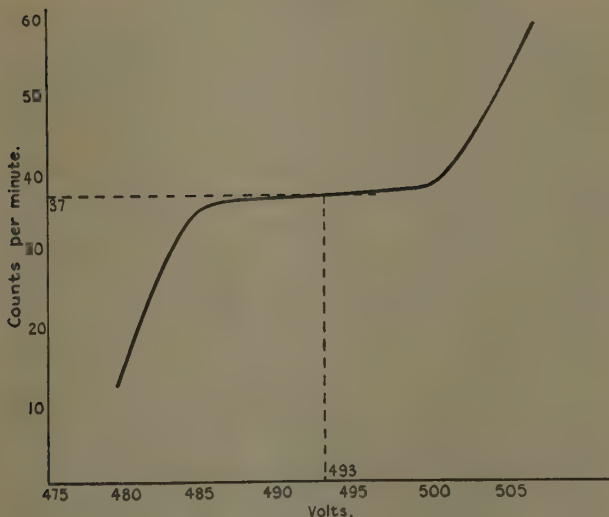
the tube could be struck between the pressures 4.5 cm. Hg and .05 mm. Hg approximately when it was connected across the oscillatory circuit, as shown by the broken lines of fig. 2. The reaction of the passing of a discharge on the oscillatory circuit was not large enough at the upper end of the pressure range to make it possible to attain a counting state—that is to say, at the higher pressures the self-restoring property required in a counting tube was absent, the discharge staying in once it was struck. At the lower end of the pressure range the self-restoring property of the system was effective, the glow taking the form of a sudden flash. The tube at pressures of the order of 0.10 mm. Hg acted as a counter quite efficiently, the natural count being determined as 9 per minute.

In order to increase the reaction on the oscillatory circuit due to the passage of a discharge, tube III. was constructed, as it was hoped that with the plates of the tube in place of the condenser C_1 of fig. 2 the self-restoring property would be present at higher pressures than those used with tubes I. and II. It was found that discharges could be obtained in tube III with air as the filling gas over a range of pressures from 0.10 mm. Hg to 7.00 cm. Hg at least. It was found also that the tube did act as a counter at the high pressure of 7.00 cm. Hg, each discharge taking the form of a very fine bright line between the plates, while at the lower end of the pressure range the discharge was a dull blue glow spread round the edges of the electrodes. The tube was usually worked at the low pressures, because in this region it was steadier in its action, the variations at pressures around 7.00 cm. Hg being possibly due to the fact that the line discharge was inclined to wander between different points on the plates. The characteristic curve for tube III. was determined with the oscillation frequency at $1.10 \cdot 10^7$ c./s. by means of an electrostatic voltmeter.

The chief difference in the case of tube III. was that the natural count was much higher than that of either of tubes I. or II. This is to be expected, since the surface area of the electrodes of tube III. and the volume of the containing vessel are larger, this giving an increased chance of radiations falling on the walls, the containing gas, and the electrodes causing the ejection of electrons etc. While considering the natural count, which is seen from fig. 6 to be 37, it is well to point out that the characteristic

curve had to be taken in the dark, or with the vessel screened from light, as this was found to increase the count by a photoelectric action. Thus a 100 watt lamp at 1 metre from the counter doubled the natural rate of counting. Experiments performed with a D.C. tube acting as a detector, the tube being an ordinary Osglim beehive lamp, showed that the photoelectric effect in these tubes was of much greater magnitude, a lighted match held near being sufficient to cause very

Fig. 6.



Characteristic curve for tube III.

rapid discharging. From the curve of fig. 6 we see that the working potential of tube III. is 493 volts, the gas pressure being .05 mm. Hg, and the voltage requires to be adjusted at this value within the limits ± 7 volts. A test carried out with tube III. to test its sensitivity to γ -rays showed that the natural count was doubled when 1 millicurie of radon was placed at 4.5 metres from the counter.

Experiments carried out with a beam of γ -rays, defined by means of lead blocks, showed that detection of the

γ -rays was effected when the γ -rays passed through the containing gas or when they impinged on the electrodes. The effect was greatest when the beam fell directly on the metal surface. It would appear that when the counters are detecting the γ -radiation most of the action is due to the γ -rays ejecting electrons from the metal, these producing ionization when they oscillate in the H.F. field of electric force. This conclusion is supported by comparison of the sensitivities of tubes I. and III. with the surface areas of the coil of tube I. and the plates of tube III.

$$\text{Thus} \quad \frac{\text{Sensitivity of tube I.}}{\text{Sensitivity of tube III.}} = \frac{3 \cdot 5^2}{4 \cdot 5^2} = 0 \cdot 61.$$

Surface area of coil of tube I. $= \pi \cdot 0 \cdot 25 \cdot 80 = 63 \text{ sq. cm.}$

Surface area of plates of tube III. $= 2\pi r^2 = 2 \cdot \pi \cdot 3 \cdot 9^2$
 $= 94 \text{ sq. cm.}$

$\therefore$ Ratio of surface areas $= 0 \cdot 67.$

Comparison of the figures 0.61 and 0.67 supports the view that the γ -rays act for the most part on the metal surfaces.

The "energy trigger ratio" of a counter is the ratio of the energy of the discharge to the energy of the ionizing particle. The energy of the discharge in the

H.F. tubes is $\int_0^T iV dt$, where i is of the order of 10 milliamps

and V of the order of 10^2 to 10^3 volts. The integral cannot be evaluated, but the effective ratio for the H.F. tubes must be large compared with that of the point-cylinder type, 10^5 to 10^9 , as determined by Appleton, Emel  us, and Barnett ⁽⁸⁾, since a small loud-speaker could be included in the circuit. The "energy trigger ratio" of the parallel plate D.C. tube may be higher than that of the point-cylinder type, but some of the faults of this type of detector mentioned above and encountered in some experiments with it make it inferior in its action to the H.F. tubes. As has been said, its action depends on polarization layers, and these give rise to erratic behaviour. For instance a relatively intense source of radon brought up close to an Osglim lamp acting as a detector sometimes stopped it working altogether. This agrees with two observations of Taylor. Taylor shows that with continued flashing polarization in some tubes can amount to values in the region of

60 volts ⁽¹³⁾. He finds moreover that a D.C. tube cannot act as a counter when the polarization is much greater than about 3 volts ⁽⁹⁾. The explanation of the above phenomenon would seem to be that the initial rapid flashing of the tube raises the polarization to considerable values, and prevents the subsequent action of the tube as a counter. Another fact noted with the D.C. tube was that after a certain maximum the counting rate could not be increased by increasing the radiation intensity. This follows from the fact that there is a time period

$$t = CR \log_e \frac{E - v_c}{E - v_b},$$

associated with the electrical circuit, where E is the potential applied to the tube, v_b is the potential at which discharge occurs when radiation falls on the detector, v_c the minimum maintenance potential. This means there is a certain minimum time interval between individual discharges, and shows that the maximum rate at which particles can be recorded is determined by the time constant CR of the circuit. If two particles enter the chamber in rapid succession the second cannot initiate a discharge unless the battery has had time to recharge the condenser to a sufficiently high voltage. In the case of the point-cylinder counter this is true also, though CR is usually smaller in value. With the H.F. discharge-tubes there is no such time constant associated with the circuit limiting the rate of counting, as was proved by the high rates of counting obtained with strong radiation intensities.

The question of the reduction of the sparking potential of the tubes was studied. A weak source of radon held near an Osgilim beehive lamp showed that the sparking potential fell from 162 volts to 159 volts. If any meaning has to be attached to such results readings have to be determined in a certain way. Considering the beehive tube, if it is put at a definite potential of 160 volts, read on an accurate electrostatic voltmeter, it will not show flashes if there is no radiation present. Keeping it at this steady potential, and bringing radon near to it, a discharge is observed between the electrodes (either continuous or flashing). The behaviour of tube II. may be contrasted with this. It was found that when

the potential across the electrodes was increased extremely slowly the discharge commenced at very nearly the same value whether radiation fell on the tube or did not. When the potential was increased more rapidly, however, there was a difference between the values at which discharge commenced of the order of 5 to 10 per cent. of the sparking potential. Also, if the tube was set at an intermediate value of potential the discharge came in after some delay in the absence of γ -radiation, but it came in as soon as radon was put near the vessel. The filling gas was argon at a pressure of 4.00 cm. Hg, and with air at a pressure of 1.31 cm. Hg the results were much the same. The same type of behaviour was found with tube I. The potential at which the discharge commenced was the same in the presence or absence of radon, the potential being increased very slowly. An increase of the order of 5 per cent. was recorded when it was raised more rapidly. For tube III., filled with air at 6.80 cm. Hg pressure, the counting commenced at very nearly the same values when the potential was increased slowly, it being a very occasional flash without radiation and rapid flashing with radon. More rapid increase gave differences of the order of 3 to 5 per cent. however. There was evidence in some cases for all the tubes of a small reduction in the sparking potential for a very slow rate of increase of potential, but it was of the order of 1 per cent. and was near the limit of accuracy of the measurements. Thus specimen readings are given in the table below (in arbitrary values only).

	Radium present.	Radium absent.
Very slow increase . . .	49.3	49.7
More rapid increase . .	49.3	52.7

The action of the radiation on the H.F. tubes seems to be essentially a reduction of the time-lag in the passing of a discharge. For instance, with tube II. which has a natural count of 9 per minute, there exists on the average a period of 7 seconds between individual counts, but this interval is reduced by the action of the radiation, becoming smaller and smaller with stronger and stronger intensities. The difference in the readings 52.7 and 49.7 is due to the rise in potential which takes place during

the 7 seconds in which no ionizing particles enter the counter, there being no radiation present except the normal cosmic radiation etc. present in any room. The different action of radiations on the H.F. discharge-tubes and on D.C. tubes was noticed in the fact that the D.C. tube could be adjusted at a potential at which no flashes took place under normal conditions although the tube flashed when γ -radiation fell on it. It was found, on the other hand, that radiation had no effect on the H.F. tube unless this was adjusted at such a voltage that an occasional flash was taking place. These results agree with those given above for the effect of γ -radiation on the sparking potential of both types of tube, and prove that the D.C. tube may be adjusted at a potential such that it detects γ -radiation but does not detect the cosmic radiation etc. present under normal conditions. For a true counter we should expect the few radiations normally present to have the same effect as strong γ -radiation and to be equally capable of affecting the counter, and this is true of the H.F. tubes described above.

A few remarks may be added here on several other points connected with the working of the H.F. discharge-tubes. The filling gas was found to have no material effect on their behaviour, though this may have been due to the fact that the purity was not very high. Commercial argon, 99.5 per cent. pure, was used, and the hydrogen was passed direct from a Hoffmann apparatus into the drying-tubes and then into the discharge-tubes. The shape of the coil of tube I. is not critical, but it was found to be the best of those tried, while the electrode disposition of tubes II. and III. may be varied considerably. Tube III., forming part of the H.F. circuit, seemed to be the best arrangement however. This brings in one of the disadvantages of this system, the fact that capacity effects from surrounding objects may alter the voltage generated across the tubes by the high-frequency oscillator and thus affect the results. This can be guarded against in practice. It may be noted that an attempt to use an electrodeless discharge, with the coil outside the containing vessel instead of inside as in tube I., proved unsuccessful. Many causes may explain this result—the long life of an ion in the glass vessel being a probable one.

The writer is much indebted to Professor E. Taylor Jones, under whose supervision the work was carried out, for his help and advice. He also thanks Dr. John Thomson for the keen interest he took in the experiments and for his many helpful suggestions. The research was performed in the Research Laboratories of the Natural Philosophy Department at the University of Glasgow.

References.

- (1) Rutherford and Geiger, *Proc. Roy. Soc. A*, lxxxi. p. 141 (1908).
- (2) Geiger and Rutherford, *Phil. Mag.* ser. 6, xxiv. p. 618 (1912).
- (3) Geiger, *Phys. Zeits.* xiv. p. 1129 (1913).
- (4) Rutherford and Geiger, *loc. cit.*; Geiger and Rutherford, *loc. cit.*; Hess and Lawson, *Wien. Ber.* cxxvii. 3 and 4 (1918); Greinacher, *Zeits. f. Phys.* xxiii. 6, p. 361 (1924).
- (5) Geiger and Muller, *Phys. Zeits.* xxix. p. 839 (1928).
- (6) Baeyer and Kützner, *Zeits. f. Phys.* xxi. 1, p. 46 (1924).
- (7) Thomson, *Phil. Mag.* ser. 7, xviii. p. 696 (1934).
- (8) Appleton, Emeléus, and Barnett, *Proc. Camb. Phil. Soc.* xxii. p. 434 (1923-25).
- (9) Taylor, *Proc. Camb. Phil. Soc.* xxiv. p. 251 (1927-28).
- (10) Taylor, *Dissertation*, Utrecht (1927).
- (11) Thomson, *Phil. Mag.* ser. 7, xxi. p. 1057 (1936).
- (12) Sir J. J. Thomson, *Proc. Phys. Soc.* xl. p. 79 (1927-28).
- (13) Taylor, *Phil. Mag.* iii. pp. 368 and 753 (1927).

XLVIII. *Optical Rotatory Power of Solutions in an Electric Field.* By J. KUNZ and S. H. BABCOCK*.

ANDREW McLEAN† has shown that a relationship exists between the rotatory power of an optically active solution and the electric moment of the solvent. It has been found that, in general, the rotation varies in the opposite sense to the electric moment of the solvent throughout a series of solvents derived from the same parent hydrocarbon. An explanation of this observation has been advanced in terms of dipolar association between solvent and optically active solute, and numerous results have been brought forward in support of this hypothesis.

These observations suggested the possibility that an electric moment induced in the solvent molecules by an external electric field might have an effect similar to that

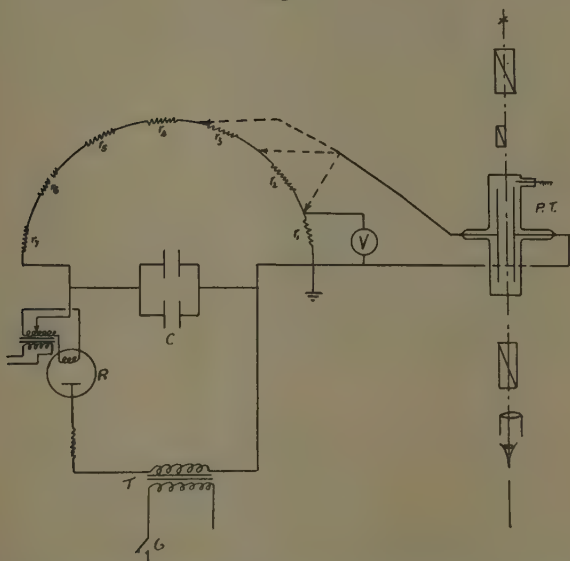
* Communicated by the Authors.

† J. C. S., 1931, pp. 674, 2652; 1932, pp. 1400, 1409, 2332; 1933, pp. 376, 1217; 1934, p. 351.

of the permanent moment. Dr. McLean and J. Kunz * in a previous note, published in 'Nature,' have described some experiments which seem to indicate such an effect. These experiments have been continued, and we wish to give an account of the method and the results.

Previous experience had shown that solvent influences are best observed with optically active compounds of simple molecular structure with a strongly polar group

Fig. 1.



or groups situated near the asymmetric centre. For this reason 2-1-menthyl-3 nitro hydrogen phthalate was considered a suitable optically active solute. Since the moment induced in the solvent molecule by an external electric field is likely to be small compared with that due to the introduction of a polar substituent, it was thought that solvents with zero or very small permanent electric moment would be most advantageous for the proposed investigation; accordingly benzene and toluene were employed as solvents.

* 'Nature,' cxxxvi. p. 795 (November 1935).

The polarimeter used was of the Lippich double field type reading to 0.01° . The source of light was a Wotan sodium lamp. The polarimeter tube employed was specially constructed for this work, and consisted of a glass cylinder 11 cm. long and 2 cm. in diameter, closed at the ends with plain parallel glass plates and furnished with two platinum electrodes 8 cm. long, 11 mm. wide, and 13 mm. apart. The cross-line in the field of the polarimeter was vertical, and to begin with the electrodes were approximately parallel to this line. When both sides of the field of vision were equally bright, and when an extra nicol prism was placed between the eye and the telescope, extinction occurred when the long diagonal of the nicol prism was vertical, *i.e.*, the plane of polarization of the polarimeter was horizontal.

The liquids in the polarimeter tube were subject to direct and alternating current fields. The general electric and optical arrangement is given by fig. 1. T is an alternating current Thordarsan transformer of 1000 watts with a secondary voltage of 25,000 volts. R is a hot filament or a Ch Knipp cold electrode rectifier. C are two parallel condensers each of 0.3 microfarad capacity. $r_1, r_2 \dots r_7$ are resistances of the following values in megohms :—

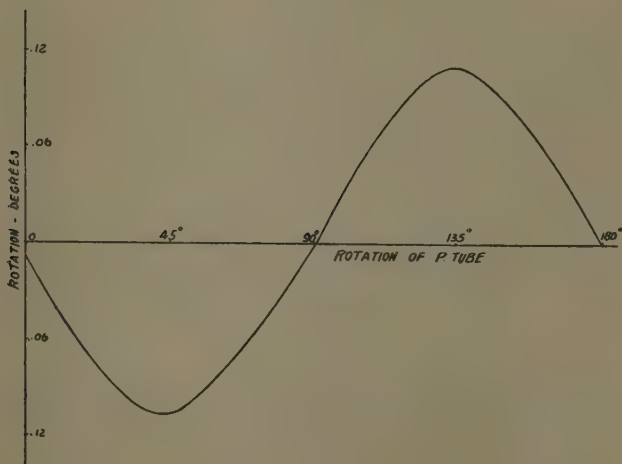
1.	2.	3.	4.	5.	6.	7.
1.95	2.12	2.01	2.05	2.20	1.88	4.7

V is an electrostatic voltmeter, reading from 500 to 2500 volts. P.T. is the polarimeter tube with thermometer, and a circular scale which could be turned about a horizontal axis. The rotation was read by means of an index rigidly fixed on the polarimeter. The nature of the rectified currents was determined by means of a cathode ray oscillograph, which gave a straight line except for insignificant deviations both for the Knipp rectifier (unpublished) and for the hot filament rectifier. From the beginning it was realized that the Kerr effect should occur under the conditions of the experiment. In order to find this effect, we filled the tube with benzene, toluene, nitrobenzene, dioctenol, carvone, limonene, and camphor dissolved in toluene, and found no effect, when

various fields up to 14,000 volts per cm. were applied and the tube turned through 45° . We concluded that with our arrangement we could not detect the Kerr effect. The Kerr effect is most marked when the plane of polarization of the incident light makes an angle of 45° with the lines of electric force; when it is parallel or perpendicular to them there is little or no effect.

We filled the tube with a 3 per cent. solution of 2-1-menthyl-3 nitrohydrogen phthalate (menthylester), applied a field of about 6,000 volts per cm., and rotated the tube from zero position—vertical line of separation

Fig. 2.

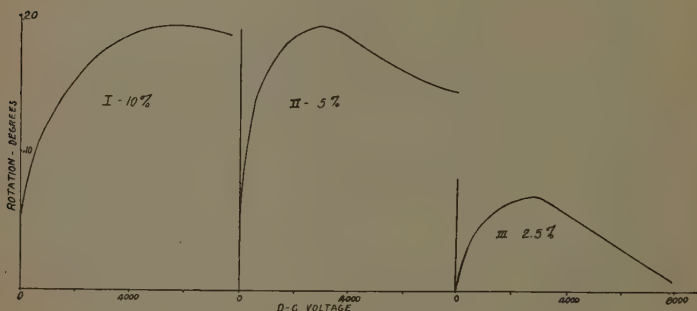


of the two sides of vision and the horizontal lines of electric force—through an angle of 180° . The results are given in the curve of fig. 2, which shows that the observed rotations coincide exactly with a sine curve. The maximum rotation was about 0.12° . The same result was obtained with other nitromenthyl benzoate. In this respect the optical effect observed was like the Kerr effect, *i. e.*, it was like elliptical polarization of light.

In order to distinguish between simple rotation of the plane of polarization and elliptic polarization the method of Rayleigh was tried. A strip of flint glass, 18 cm. long, of rectangular cross-section 9 mm. high and 12 mm. wide,

was placed between the analyser and the polarizer. It was supported at both ends and stressed by a wedge acting on the middle of the strip. When the nicols were at right angles, and pressure was applied, the field lighted up and could not be reduced to darkness by the rotation of the analysing nicol. But, unfortunately, the dark line, due to the neutral filament in the stressed glass rod, could not be seen, and the question could not be decided. It seemed though that darkness could be realized again when the glass rod was replaced by menthylester in the polarimeter tube under the action of an electric field.

Fig. 3.



Previous observations having shown, however, that the Kerr effect was not present, we proceeded to investigate the effect of varying the field strength at a constant angle of 45° with the plane of polarization. The Kerr effect is independent of the direction of the electric force, and its magnitude is proportional to the square of the electric intensity. It was therefore important to determine the rotation as function of the field strength.

A Knipp rectifier was used and the potential was varied through the various resistances $r_1, r_2 \dots r_7$ of fig. 1 from 500 to 8000 volts. The curves of fig. 3 were obtained for solutions of 2.5, 5, and 10 per cent. menthylester in benzene. The deflexion (rotation) increases with increasing field strength, reaches a maximum and decreases with increasing field. This effect has been observed in each case, and could easily be repeated.

Sometimes, however, it seemed to become constant with increasing potential difference. These curves show the distinct differences between our effect and the Kerr effect. In the first measurement we observed a strong after effect, which, however, vanished as soon as the condensers were discharged. Nevertheless, there was sometimes a small time effect observed, for instance when the switch of fig. 1 was closed the deflexion increased in the first 45 seconds and then assumed a constant value.

The compounds with the menthylester group rotate the plane of polarization to the left. When the field in the tube was turned anti-clockwise through an angle of 45° the natural rotation decreased. It was also observed that for solutions of the same concentration with respect to the ester the rotation decreases markedly as the electric moment of the solvent increases. This effect has been found for benzene and toluene solutions of the same ester. The moment induced in the solvent molecules by an external electric field also lowers the rotation when the angle between the direction of the field and the plane of polarization of the light is increased by rotating the tube in an anti-clockwise direction. But by a rotation of the tube in a clockwise direction the natural rotation was increased.

The rotation was independent of the direction of the electric field.

When the alternating current was rectified with a Knipp rectifier, but without the condenser, the current had the form of fig. 4 *a*, and the 2.5 per cent. solution of menthyl ester in benzene showed a strong hysteresis effect, shown by the curves of figs. 4 *b* and 4 *c*. This effect was enhanced by an alternating current field, which was obtained by removing the Knipp rectifier and the condenser of fig. 1. The rotation here obtained with a 2.5 per cent. of menthylester in benzene rose to about 0.5° with alternating currents. The first curve is given by fig. 5, *a*; the next curve, fig. 5, *b*, summarizes the results obtained immediately afterwards by means of a direct current with the condenser included. These curves show how, under the influence of a direct field, the rotation gradually decreased. After these readings the solution was left without a field for 12 hours whereupon a direct field was applied again; at first

Fig. 4.

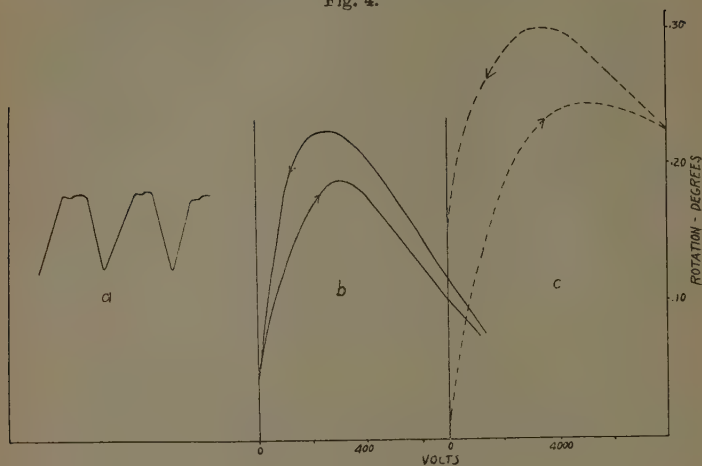
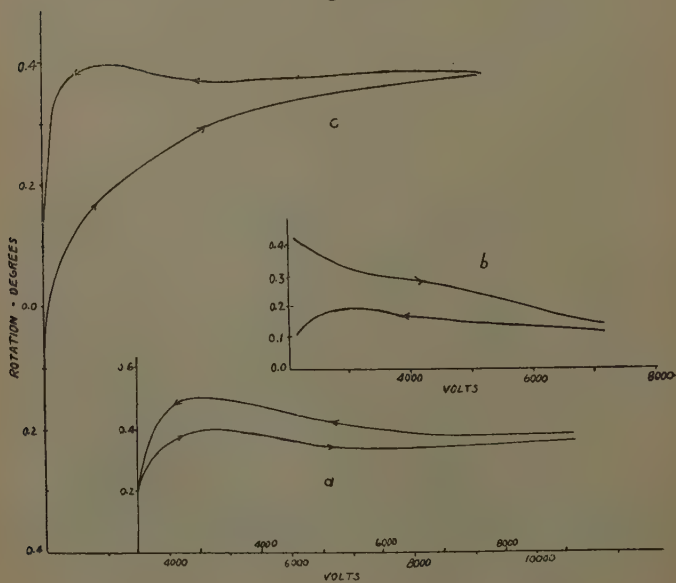
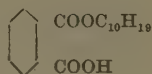
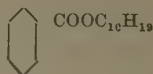
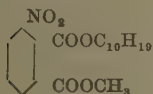


Fig. 5.



the rotation had increased from 0.15° to 0.34° , but decreased again to 0.10° with increasing potential difference, and then increased slightly to 0.15° with decreasing field. Finally, under the action of an increasing A.C. field, the deflexion rose from 0.11° to 0.4° at 14,400 volts, and showed strong hysteresis with decreasing field strength as fig. 5, c, shows.

These optical properties of menthylester were the same for solutions in ordinary and in very dry benzene. Besides the menthylester the following menthyl compounds were investigated :—



dissolved in benzene (5 per cent. solution). These substances showed the same phenomena as menthylester, but in a smaller degree, the last acid gave the smallest effect, the second substance the largest rotation of the three compounds.

Finally, a solution of 2-ethyl-3-nitrophthalate in toluene was submitted to an electrostatic field in the polarimeter tube. This compound is optically inactive, and much less soluble in benzene and toluene than the corresponding menthylester. The solution was made by heating the ester and toluene and, while hot, poured into the tube. The curve obtained resembles the curve III of fig. 3, the tubing being rotated 45° in an anti-clockwise direction. It would seem that the rotatory power is again a function of the potential difference, reaching a maximum at about 3000 volts with both direct and alternating electric fields. It was very difficult to repeat these results, as the solution seemed to change continuously during the experiment. With the same field strength the polarimeter readings were very erratic, *e.g.*, at one observation the application of a field of 14,300 volts caused an instantaneous rotation of 3.01° , but in a few seconds, whilst the field was still on, it dropped to 0.35° . After the solution had been standing for three days even the most powerful field had no effect on it; this was probably because most of the solute had crystallized out. However, with a fresh, hot solution rotational changes were again observed. The effects obtained

with the ethylester appear to be distinct from those exhibited by the menthylester. We heated the polarimeter tube by an electric current, but without success.

It is known that ions in solutions have an effect on the rotatory power. The resistance of the various substances here studied was, therefore, measured in order to get an idea of the conductivity. The menthylester in the polarimeter tube had a resistance of $5,34 \cdot 10^9$ ohms; the other menthyl compounds showed resistances of the order of magnitude 10^{11} ohms.

It is hoped to continue these investigations along experimental and theoretical lines.

The authors wish to thank the Chemistry Department of the University of Illinois for the use of the polarimeter, Professor Charles T. Knipp for the rectifier, Professor R. L. Shriner for the chemicals, and Professor J. T. Tykociner for the high resistances and the voltmeter.

Summary.

Various optically active menthyl compounds have been subjected to an electric field. No Kerr effect has been observed, but the rotation of the plane of polarization depends on the angle between that plane and the lines of the external electric field. The rotation of the plane of polarization is a function of the field strength quite different from the Kerr effect.

University of Illinois,
Urbana, Illinois.
January 28, 1936.

XLIX. *Fundamental Physical Constants.* By W. N. BOND, M.A., D.Sc., F.Inst.P., Lecturer in Physics in the University of Reading*.

SUMMARY.

ESTIMATES of the numerical values of fundamental physical constants are deduced from Eddington's theories concerning M/m , $hc/2\pi e^2$ and the universal gravitational constant, G . Confirmatory experimental evidence is given, and the remaining discrepancies are pointed out. The

* Communicated by the Author.

paper concludes with Tables of data and deductions, and a fairly extensive bibliography.

Introduction.

SINCE about 1929 Sir Arthur Eddington has developed theories which predict comparatively simple numerical relationships between certain of the fundamental physical constants (*e. g.*, the electronic charge and Planck's constant). A considerable part of the previous advances in physical theory had consisted in the discovery of such relationships; and there can be little doubt that relationships remained to be discovered (and that others still remain). Relationships had been suggested by various workers ^{(19), (46)}; but Eddington's results have a much more complete theoretical basis, correlate more data, and agree more closely with experiment than do the results of the other workers.

Prof. H. S. Allen has suggested to me that it would be useful to publish a short table giving the best values of the fundamental physical constants consistent with Eddington's theories. In this paper I am giving such a table, a brief discussion of some of the numerical results, and a fairly extensive bibliography of the papers relating to the subject.

The Values of M/m and e/m .

According to Eddington ⁽⁹⁾, the ratio of the rest-masses of proton and electron, M/m , is equal to the ratio of the roots of the equation

$$10x^2 - 136x + 1 = 0.$$

This gives $M/m = 1847.599_5$.

Assuming this result, Birge ⁽²⁴⁾ deduced the corresponding value of the specific electronic charge to be

$$e/m = (1.770, 3_1 \pm 0.000, 14) \times 10^7 \text{ (abs. e.m.u.)}.$$

This is in disagreement with the recent accurate experimental determinations, which give ^{(37), (28)}

$$(1.757, 6_4 \pm 0.000, 2) \times 10^7.$$

I suggested ⁽³⁶⁾ that the experiments were really measurements of

$$\frac{136}{137} (1.770, 3_1 \pm 0.000, 14) \times 10^7,$$

that is to say, of

$$(1.757,4 \pm 0.000, 14) \times 10^7.$$

Eddington ⁽¹³⁾ said that it was likely that the ratio of the experimental and theoretical estimates of e/m would be very nearly equal to $\frac{136}{137}$. According to

Eddington, we may expect to measure $(M+m)$ correctly, but to obtain a value for $Mm/(M+m)$ that is too large by a factor $\frac{137}{136}$. Hence, since M is much larger than m , we obtain almost the correct value for M , but too large a value for m . In fact we should have

$$M = (1.000,004) (M \text{ experimental}),$$

$$m = \frac{136}{137} (1 - 0.000,004) (m \text{ experimental}),$$

$$M/m = \frac{137}{136} (1.000,008) (M/m \text{ experimental}).$$

The experimental and theoretical evidence support my hypothesis that the experimental value of e/m is $\frac{136}{137}$ of the theoretical value. It might be convenient to term the theoretical value the "true" value and the experimental value the "apparent" value.

The Values of e and h .

According to Eddington's theories ⁽⁴⁾ we have the relationship

$$\frac{hc}{2\pi e^2} = 137,$$

where h = Planck's constant, c = velocity of light in a vacuum, and e = electronic charge (in e.s.u.).

The velocity of light was estimated by Birge ⁽²⁰⁾ as

$$(2.99796 \pm 0.00004) \times 10^{10} \text{ cm./sec.}$$

(Michelson's experiments), Rosa and Dorsey's indirect method giving

$$2.9979 \pm 0.0001.$$

The later experiments ⁽⁴⁰⁾ give the value

$$2.99774,$$

with a probable error of about ± 0.00002 . These later experiments show evidence of small systematic errors, and disagree somewhat with the previous results. Hence I will adopt as the likely value

$$c = (2.9978_5 \pm 0.0000_5) \times 10^{10} \text{ cm./sec.}$$

In order to obtain values for e and h , one further relationship is required. Assuming the Bohr-Rydberg theory, the value of Rydberg's constant for an infinitely massive nucleus ⁽²⁰⁾, Eddington's $M/m = 1847.60$, the value of Faraday's electrolytic constant ⁽²⁰⁾, and the value of the atomic weight of H^1 (relative to the normal mixture of O^{16} and O^{18} as 16) ⁽²⁴⁾, we obtain

$$e = (4.775_7 \pm 0.0004) \times 10^{-10} \text{ abs. e.s.u.,}$$

$$h = (6.549 \pm 0.001) \times 10^{-27} \text{ erg. sec.}$$

The result of Millikan's work on the value of e has been estimated by Birge ⁽²⁰⁾ and by Millikan ⁽⁴¹⁾ respectively as

$$4.768 \pm 0.005,$$

$$4.770 \pm 0.005.$$

A recent determination, based on the automatic counting of α -particles ⁽⁴²⁾, gives

$$4.768 \pm 0.005.$$

These values are in reasonable accord with the result deduced from Eddington's theories (though rather lower than that result).

On the other hand, by using X-rays to compare the spacing of atoms in a crystal with the spacing of a ruled grating, the following estimates have been obtained ^{(37), (27)}:

$$4.806 \pm 0.003,$$

$$4.805,$$

$$4.8036 \pm 0.0005.$$

I suggested ⁽³⁷⁾ that these were really estimates of

$$\frac{137}{136} (4.775_7 \pm 0.0004) \times 10^{-10}$$

$$= (4.810_7 \pm 0.0004) \times 10^{-10} \text{ abs. e.s.u.}$$

This hypothesis would account for most of the discrepancy (though, again, the experiments give rather lower values than those predicted).

I believe that Eddington thinks that the factor $\frac{136}{137}$ would be expected to occur here also. The X-ray method

of estimating e makes use of equations of the general form

$$e = \frac{(\text{charge per gram of crystal})(\text{grams in a unit cell})}{\text{number of electronic charges in a unit cell}},$$

(grams in a unit cell) = (density of crystal)(volume of a unit cell).

When we speak of the mass contained in a single unit cell, we are treating one of a large number of indistinguishable cells as if it were separately identifiable with certainty. It would seem that measurements of its mass will yield a value that is too large by a factor that is very nearly

$\frac{137}{136}$ (as in the case of the electron). Hence we ought

to write :—

$$(\text{grams in a unit cell}) = \frac{136}{137} (\text{density of crystal})(\text{volume of a unit cell}).$$

If these suppositions are correct, the X-ray wave-lengths as measured by a ruled grating would be based on correct theory. But the wave-lengths deduced from measurements with a crystal, assuming the value of e ,

would be too small by a factor $\sqrt[3]{\frac{136}{137}}$; that is to

say, they require increasing by 0.245 per cent.

An alternative explanation of the discrepancy between the two sets of estimates of e has been given ⁽²⁷⁾, ⁽³⁸⁾. It has been suggested that Millikan assumed too small a value for the viscosity of air. The chief fresh information on which this suggestion is based is the viscosity determination made by Kellström ⁽⁴³⁾. It seems to me inadvisable to adopt the value $\mu_{23} = (1834.8 \pm 3.0) \times 10^{-7}$, based on a single recent determination, in place of Millikan's estimate of $\mu_{23} = (1822.7 \pm 1) \times 10^{-7}$, which was based on experimental work by several workers, extending over a period of some years.

The Values of σ , c_2 , etc.

Continuing to assume the validity of Eddington's theories, we can deduce estimates of the Stefan-Boltzmann constant, σ , and the second radiation constant, c_2 :—

$$\sigma = (5.737_5 \pm 0.002_5) \times 10^{-5} \text{ erg. cm.}^{-2} \text{ deg.}^{-4} \text{ sec.}^{-1},$$

$$c_2 = (1.4303 \pm 0.0002) \text{ cm. deg.}$$

These are in excellent agreement with the experimental values as given by Birge ⁽²⁰⁾, ⁽²⁴⁾ :—

$$\sigma = (5.735 \pm 0.011) \times 10^{-5},$$

$$c_2 = 1.432 \pm 0.003.$$

The measurements on the photo-electric effect that Birge ⁽²⁰⁾ considered the most accurate (those made by Lukirsky and Prilezaev) give

$$h/e = (1.371_6 \pm 0.003) \times 10^{-17} \text{ e.s.u.},$$

whereas the value deduced from Eddington's theories is

$$h/e = (1.3712 \pm 0.0001) \times 10^{-17}.$$

Again the agreement is excellent.

The ionization potential data for mercury that were used by Birge ⁽²⁰⁾ give the value

$$h/e = (1.375_1 \pm 0.003) \times 10^{-17},$$

which is also in quite satisfactory agreement with the predicted value.

Results deducible from measurements of the limit of X-ray spectra, and from other recent data ⁽²⁷⁾, ⁽²⁸⁾, do not seem to agree with the values I have suggested. If these data are included, however, there are inconsistencies, no matter what values be adopted for e and h . I can only conclude that these data are affected by errors of experiment or theory, or both. It is quite likely

that factors such as $\frac{136}{137}$ and $\sqrt[3]{\frac{136}{137}}$ may be involved, but it does not seem possible to clear up the subject as yet.

In the case of some of these data, agreement is obtained by using the crystal and grating estimate of e together with the experimental value of e/m —that is to say, by omitting to take the factor $\frac{136}{137}$ into account in either case ⁽⁴⁴⁾.

The Value of G.

In Eddington's most recent papers on these subjects ⁽¹⁴⁾, ⁽¹⁵⁾ he develops a theory according to which the number of

particles in the universe N , is related to the value of the universal gravitational constant G by the equation

$$\sqrt{\frac{10}{3}} N = \frac{136}{10} \frac{hc}{2G(M+m)^2}.$$

TABLE I.

Experimental data assumed.		References.
Velocity of light, C	$= (2.9978_5 \pm 0.0000_5) \times 10^{10}$ cm./sec.	(20) & (40)
Atomic weight, H^1	$= 1.00756 \pm 0.00005$	(24)
Faraday's constant, F	$= 9648.9 \pm 0.7$ abs. e.m.u./gm.-equiv.	(20)
Rydberg's constant, R_∞	$= 109737.42 \pm 0.06$ cm. $^{-1}$	(20)
Gas constant R_0	$= (8.3136 \pm 0.0010) \times 10^7$ ergs/deg. gm.-mol..	(20)

TABLE II.

Deductions from Eddington's theories.	Experimental values referred to in this paper.
$M/m = 1847.599_5$.	
$e/m = (1.770,3 \pm 0.000,1_4) \times 10^7$ abs. e.m.u.	
$(136/137)(e/m) = (1.757,4 \pm 0.000,1_4) \times 10^7$ abs. e.m.u.	$1.757,6 \pm 0.000,2$
$hc/2\pi e^2 = 137$.	
$e = (4.775_7 \pm 0.0004) \times 10^{-10}$ abs. e.s.u.	$\left\{ \begin{array}{l} 4.769 \pm 0.005 \\ 4.768 \pm 0.005 \end{array} \right.$
$(137/136)e = (4.810_7 \pm 0.0004) \times 10^{-10}$ abs. e.s.u.	$\left\{ \begin{array}{l} 4.806 \pm 0.003 \\ 4.805 \\ 4.8036 \pm 0.0005 \end{array} \right.$
$h = (6.549 \pm 0.001) \times 10^{-27}$ erg. sec.	
$h/e = (1.3712 \pm 0.0001) \times 10^{-17}$ erg. sec./e.s.u.	$\left\{ \begin{array}{l} 1.371_6 \pm 0.003 \\ 1.375_1 \pm 0.003 \end{array} \right.$
$\sigma = (5.737_5 \pm 0.002_5) \times 10^{-5}$ erg. cm. $^{-2}$ deg. $^{-4}$ sec. $^{-1}$	5.735 ± 0.011
$c_2 = 1.4303 \pm 0.0002$ cm. deg.	1.432 ± 0.003
$G = (6.658_3 \pm 0.001_2) \times 10^{-8}$ cm. ^{3} gm. $^{-1}$ sec. $^{-2}$	$\left\{ \begin{array}{l} 6.6576 \pm 0.001 \\ 6.65786 \pm 0.0017 \\ 6.664 \\ 6.674 \\ 6.678 \end{array} \right\} 6.67_0 \pm 0.00_5$

Using this equation, and taking the value of G adopted by Birge ⁽²⁰⁾ in 1929, ($G = 6.664 \times 10^{-8}$), Eddington finds

$$N = 1.5727 \times 10^{79} = 135.82 \times 2^{256}.$$

After discussion, he remarks ⁽¹⁴⁾ that "it seems probable that the exact value of N is 136×2^{256} ." If we assume

that N has exactly this value, the calculation may be reversed, giving an estimate of G :

$$G = (6.658_8 \pm 0.001_2) \times 10^{-8} \text{ cm.}^3 \text{ gm.}^{-1} \text{ sec.}^{-2}$$

The value Heyl obtained with platinum balls (the value used by Birge (1929) and by Eddington) was

$$6.664 \pm 0.002.$$

In Heyl's later work ⁽⁴⁵⁾ he obtained for glass, and for gold,

$$6.674 \pm 0.002,$$

$$6.678 \pm 0.003.$$

These are all rather higher than the value deduced from Eddington's theory. Heyl's results, however, differ from one another by more than can be accounted for by their probable errors ; and special tests appeared to show that the differences were not due to difference in the properties of the three materials used.

The values that were formerly regarded as the most accurate (see Ency. Brit. 10th ed. xxv. p. 739) were those of Boys and of Braun, namely

$$6.6576 \pm 0.001$$

and

$$6.65786 \pm 0.0017.$$

These are in excellent agreement with Eddington's theory.

Notes added on May 5th :—

(1) A fresh determination of e by the oil-drop method has been made (E. Bäcklin and H. Flemberg, 'Nature,' cxxxvii. p. 656, 1936) giving a slightly lower value than that obtained by Millikan.

(2) Schopper ⁽⁴²⁾ subsequently detected irregularities in his standard cell and decided to repeat his experiments. (*Z. Phys.* xciv. p. 649, 1935). Schopper's original result was criticised by R. T. Birge and McMillan (*Phys. Rev.* xlvii. p. 320, 1935).

(3) I have recently measured the viscosity of dry air by a new capillary tube method and obtained the value

$$\eta_{23} = (1833.8 + 0.8) \times 10^{-7}.$$

This confirms Kellström's result ⁽⁴³⁾ and indicates that the X-ray and oil-drop methods of estimating e both measure

what I have termed $\left(\frac{137}{136}e\right)$

(4) Aston has obtained and confirmed a slightly different value of H^1 from that given in Table I. (p. 630) (F. W. Aston, 'Nature,' cxxxvii. p. 357 (1936); M. L. Oliphant, 'Nature,' cxxxvii. p. 397, 1936).

Bibliography.

- (1) A. S. Eddington, Proc. Roy. Soc. A, cxvii. p. 358 (1929).
- (2) —, 'Nature,' cxxiv. p. 840 (1929).
- (3) 'Nature,' cxxiv. p. 320 (1929).
- (4) A. S. Eddington, Proc. Roy. Soc. A, cxxvi. p. 696 (1930).
- (5) 'Nature,' cxxvi. p. 942 (1930).
- (6) A. S. Eddington, Proc. Camb. Phil. Soc. xxvii. p. 15 (1931).
- (7) —, Proc. Roy. Soc. A, cxxxiii. p. 605 (1931).
- (8) —, Proc. Roy. Soc. A, cxxxiii. p. 311 (1931).
- (9) —, Proc. Roy. Soc. A, cxxxiv. p. 524 (1931).
- (10) —, Proc. Phys. Soc. xlv. p. 1 (1932).
- (11) —, Proc. Roy. Soc. A, cxxxviii. p. 17 (1932).
- (12) —, Proc. Roy. Soc. A, cxliii. p. 327 (1934).
- (13) —, 'Nature,' cxxxiii. p. 907 (1934).
- (14) —, Roy. Ast. Soc. M. N. xcv. p. 636 (1935).
- (15) —, Proc. Roy. Soc. A, clii. p. 253 (1935).
- (16) G. Temple, 'Science Progress,' xxvii. pp. 396–404, 571–573 (1933).
- (17) —, Phys. Soc. 'Reports on Progress in Physics,' p. 1 (1934).
- (18) —, Proc. Roy. Soc. A, cxlv. p. 344 (1934).
- (19) R. Fürth, *Phys. Zeit.* xxx. p. 895 (1929).
- (20) R. T. Birge, 'Reviews of Modern Physics,' i. p. 1 (1929).
- (21) —, Phys. Rev. xxxv. p. 1015 (1930).
- (22) —, Bulletin of American Phys. Soc., Dec. 8th, 1931.
- (23) —, Phys. Rev. xl. p. 228 (1932).
- (24) —, Phys. Rev. xl. p. 319 (1932).
- (25) —, 'Nature,' cxxxiii. p. 648 (1934).
- (26) —, 'Science,' lxxix. p. 438 (1934).
- (27) —, Phys. Rev. xlviii. p. 918 (1935).
- (28) —, 'Nature,' cxxxvii. p. 187 (1936).
- (29) W. N. Bond, 'Nature,' cxxiv. p. 408 (1929).
- (30) —, Phil. Mag. x. p. 994 (1930).
- (31) —, 'Nature,' cxxvii. p. 164 (1931).
- (32) —, 'Nature,' cxxvii. p. 557 (1931).
- (33) —, Phil. Mag. xii. p. 632 (1931).
- (34) —, Proc. Phys. Soc. xlv. p. 374 (1932).
- (35) —, Phys. Rev. xli. p. 368 (1932).
- (36) —, 'Nature,' cxxxiii. p. 327 (1934).
- (37) —, 'Nature,' cxxxv. p. 825 (1935).
- (38) K. Shiba, Sci. Papers Inst. Phys. & Chem. Res. Tokyo, xxi. p. 128 (1933).
- (39) H. R. Robinson, Phys. Soc., "Reports on Progress in Physics," p. 247 (1935).
- (40) A. A. Michelson, F. G. Pease, and F. Pearson, *Astrophys. J.* lxxxii. p. 26 (1935).
- (41) R. A. Millikan, Phys. Rev. xxxv. p. 1233 (1930).
- (42) E. Schopper, *Zeits. f. Physik*, xciii. p. 1 (1934).
- (43) G. Kellström, 'Nature,' cxxxvi. p. 682 (1935).
- (44) A. E. Ruark, Phys. Rev. xlvii. p. 316 (1935).
- (45) P. R. Heyl, 'Bureau of Standards Journal of Research,' Dec. 1930.
- (46) G. N. Lewis and E. Q. Adams, Phys. Rev. iii. p. 92 (1914).

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The University of Reading.
Feb. 7th, 1936.

L. *Experimental Investigation of the Effects of Pressure and Current on the Intensities of the Balmer Lines of Hydrogen.* By W. W. JACKSON, M.Sc. (Manchester), Ph.D. (Aberdeen), Carnegie Teaching Fellow in Natural Philosophy, University of Aberdeen *.

[Plate III.]

Introduction.

THE conditions in a hydrogen discharge-tube can be controlled so as to give rise to an emission of intense atomic lines over a large portion of the tube, and therefore in this part there is an assembly consisting of (a) neutral atoms, some in the ground state and others in higher quantum states, (b) protons, (c) electrons, (d) quanta of different known energies.

The intensity of any one line depends on the number of atoms which change spontaneously from the higher to the lower quantum state of the two states involved for that particular line. All transitions between the two states need not contribute to the intensity, for in some collisions of any one atom, the energy of transition can go to the other colliding particle. If, however, we knew the number of atoms in the different states at any time, then, since the relative values of the transition probabilities are known, we ought to be able to predict the relative intensities of say H_α and H_β , and a complete knowledge of the assembly ought to lead to a prediction of the detailed behaviour of the discharge. Conversely, by examining the relative intensities of the lines under different conditions, we might get some information not only about the assembly itself but also about the mechanisms involved in the atomic changes taking place.

With this end in view, previous measurements † have been extended in this paper, with improved design of apparatus.

Description of Apparatus.

The apparatus differs from the earlier form as follows :—

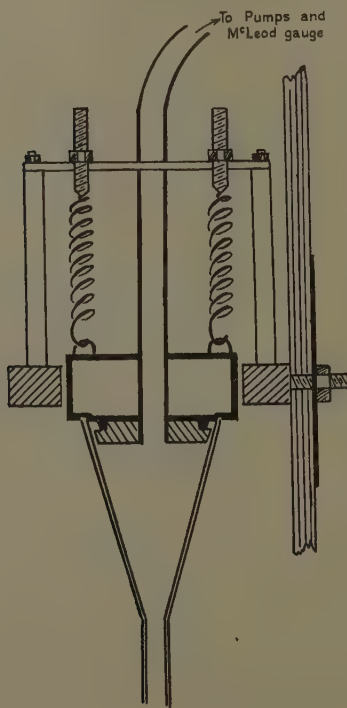
1. *Discharge-tube.*—A new tube was used, of fused silica, 1 metre in length with provision for air cooling, and with

* Communicated by the Author.

† W. W. Jackson, *Phil. Mag.* xvii. p. 33 (1934).

water-cooled electrodes faced with aluminium, sealed to the tube with a rubber ring and Apiezon sealing compound. With these cooling arrangements the tube would carry 700 m.a. at 1100 volts for five hours. Fig. 1 shows how the tube was relieved from strain.

Fig. 1.



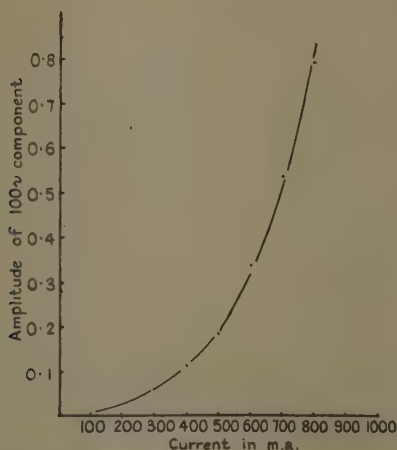
Tube support.

2. *High Tension Supply.*—A cathode ray oscillograph was used to examine the characteristics of the tube when it was excited by a double wave rectifier without smoothing, and the results throw light on the behaviour of a discharge-tube at varying currents as well as showing that a well-smoothed D.C. supply is necessary for accurate analysis of the conditions in the discharge.

The H.T. supply was built along the usual lines and gave a smoothed D.C. output up to 1 amp. at 1100 volts. The amplitude of the remaining 100~ per sec. ripple was measured and is shown in fig. 2.

3. *Spectrograph*.—Using the Hilger quartz E. 1 spectrograph and hypersensitized Q-type plates the Paschen series was not registered after 25 hours' exposure at 700 m.a. Arrangements are in hand to use the Z-type of plate, and a spectrograph of larger numerical aperture has been constructed. The spectrograph * used in the

Fig. 2.



The amplitude of the 100~ component expressed as percentage of discharge current.

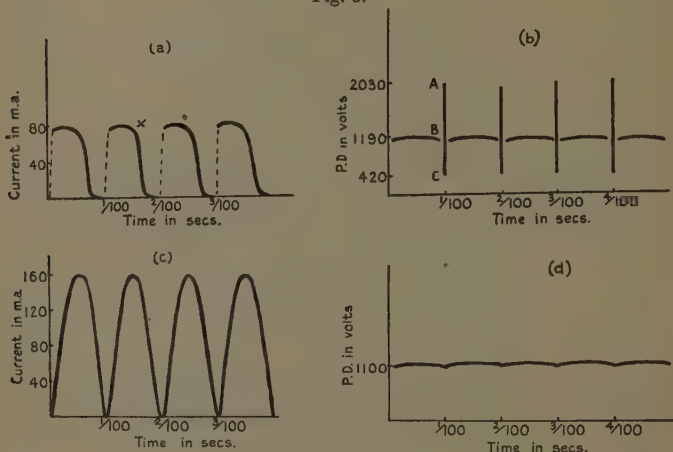
following measurements had an aperture of F/7, and the whole aperture was used, the image of the slit in any one wave-length being focussed on to the curved slit of the photo-cell carrier, which could move so as to keep the slit in the focal plane of the camera. Comparing the two spectrograms, one of 40 mins. exposure and the other of 2 mins. exposure, it is seen that the intensity of the background is negligible, so that the slit can be made

* I am indebted to Professor J. A. Carroll for permission to use and modify this instrument, which was designed for use in the 1933 Solar Eclipse Expedition.

slightly wider than the image and a survey made of the line (Pl. III.). Local variations in the response of the sensitive layer of the Weston Photo-cell were of no importance, as the radiated area was about 5 sq. cm. The external resistance to the cell was of the order of 1 per cent. of that of the cell and the current measured about 10^{-8} amp. for H_{α} , so that the response of the cell could be taken as linear.

4. *Galvanometer*.—A Zernicke galvanometer with rigid, shielded leads was used, and a shunt of copper wire with

Fig. 3.



Typical oscillograms.

amalgamated mercury contacts enabled H_{α} to be compared with the other lines as described later.

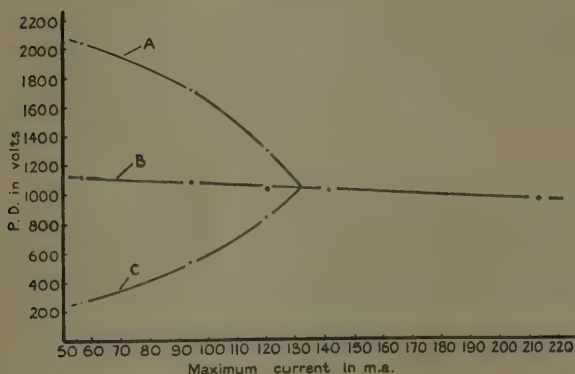
5. *Standard Lamp*.—This was of the tungsten ribbon type, and the current through it could be brought to within 1 part in 10,000 of the specified current.

Oscillograph Results.

In order to illustrate the forms of the oscillograms two specific cases are taken, and the diagrams show the observed oscillograms. In both cases the pressure in the discharge-tube is 0.41 mm. and fig. 3 (a) and (b)

indicate the current-time and voltage-time relations when the maximum value of the current is 80 m.a. It is to be observed that there is a break in the current, and the voltage-time diagram shows the sudden high kick of P.D. before the discharge starts again, whereupon the voltage drops and comes to a steady value until the next break of the current occurs. If, however, the value of the maximum current was raised to 160 m.a. the oscillograms took the form in fig. 3 (c) and (d) showing no break in the current and no high kicks of P.D. The change from the first set of conditions to the second set was investigated by making use of the experimentally

Fig. 4.



Graph of P.D. against maximum current.

determined calibration graphs for the oscillograph and measuring the P.D. corresponding to the points A, B, C and the current corresponding to X. By plotting the values of A, B, C in volts against the current X in m.a. the graph in fig. 4 was obtained and shows that the breaks in the current only exist for maximum currents less than 130 m.a. for the frequency used, which is such that the time between consecutive maxima of the current is $1/100$ sec.

An explanation may be put forward in terms of the coefficient of recombination of the ions. If, for currents whose maximum value is greater than 130 m.a., the rate of recombination of the ions is less than the rate of

decrease of the current as it approaches a zero value, then there will still be some ions present to maintain the discharge for the next cycle. On the other hand, for currents of maximum value less than 130 m.a. the rate of recombination may be greater than the gradient of a sine curve as it approaches zero value from the maximum of less than 130 m.a., the time interval between zero values being $1/100$ sec., and we might expect the discharge to go out, in which case the voltage will rise rapidly and the discharge will strike again, causing a drop of the P.D. Thus according to the above considerations we might be able to calculate an approximate value of the coefficient of recombination by finding the rate of decrease of the number of ions as the current approaches zero from its maximum value of 130 m.a.

The current in m.a. is expressed by $C = |130 \sin 100\pi t|$, and by assuming an E.S. field of between 5 and 50 volts/cm. and a mean free path of $\frac{1}{6}$ cm. for the hydrogen ion at $\frac{1}{2}$ mm. pressure, the value of the average velocity of an ion given by $\frac{1}{2} \sqrt{\frac{Xe\lambda}{2m}}$ is between 3×10^5 and 1×10^6 cm./sec., and the number of ions per c.c. to carry the current of 130 m.a. will need to be x , where x is between 2.4×10^{12} and 8×10^{11} . The number of ions carrying the current then must be represented by $|x \sin 100\pi t|$.

The gradient of this curve at $t = \frac{1}{100}$ /sec. will be $-x 100\pi$.

That is to say, as the curve approaches its zero value the rate of decrease of the number of ions necessary to carry the current is $100\pi x$ per c.c./sec., and according to the above conjectures this will be equal to the rate of recombination.

At a time 10^{-4} sec. before the zero value the number of ions is $\pi x/100$, and these recombine in 10^{-4} sec.

Since $\frac{dn}{dt} = -\alpha n^2$ we get α lies between 10^{-7} and 4×10^{-7} .

The value of α is 1.4×10^{-6} at atmospheric pressure and decreases to $1/4$ of this value at $1/8$ of an atmosphere, so that the calculated α is just about the right order of magnitude, and thus there is some basis for regarding the above theory as a probable explanation of the observed facts. The value of α is probably less than the value calculated

above, since to maintain the discharge there will need to be some ions left uncombined.

Method of Measuring Intensities.

Throughout the measurements, the light from the side of the central portion of the tube has been used, where the intensity of the background is negligible compared with the intensity of any of the lines under investigation. The galvanometer was read to 0.1 mm., and a shutter in front of the spectrograph slit was then opened and the galvanometer read again, allowing plenty of time for it to come to a stationary value. Then the shutter was closed and the zero reading taken again. This sometimes differed by as much as 1 mm. from the first zero reading and the mean of the two was subtracted from the deflected reading, and this was taken as the deflexion corresponding to the intensity of the particular line under investigation at the conditions obtaining at the time. There was no observable deflexion when the shutter was opened with no current through the discharge-tube. To cover the range of intensities measured, a specially made shunt was used. The effects of contact potentials were eliminated, and the scale of the galvanometer deflexion was calibrated and showed that the angular deflexion was proportional to the current. Whenever any change was made in the current through the discharge-tube, it was found necessary to allow some 10 minutes to elapse, in order that the tube could settle down to its new equilibrium condition, owing to the temporary change in pressure.

The standard lamp was used to calibrate the energy response of the photo-cell, the light from the central portion of the tungsten ribbon was focussed on to the slit of the spectrograph, using the same condensing system as for the hydrogen spectrum, and the experimentally determined constants of the spectrograph enabled the calibration to be effected in the usual manner. The photo-cell was calibrated at two temperatures differing by a much greater amount than any change in temperature occurring during the taking of the results, and it was found that one calibration could be used over the small temperature variation which occurred whilst the results were being taken.

RESULTS.

Section I.

Variation of Intensity of each Individual Line with Increasing Current.

Table I. and figs. 5 and 6 give the angular deflexions produced by the four Balmer lines H_{α} , H_{β} , H_{γ} , H_{δ} when the current was varied at the same constant pressure. In all diagrams the observations are represented by dots.

The figures in Table I. are not to be used for the inter-comparison of line intensities, only the figures in any one vertical column being intercomparable. They could

TABLE I.

Variation of Individual Lines with varying Current at a Constant Pressure of 0.38 mm. of Hg.

Current in m.a.	Deflexions in cm.			
	H_{α} .	H_{β} .	H_{γ} .	H_{δ} .
100	2.9	0.69	0.62	0.69
200	6.2	1.43	1.24	1.41
300	10.0	2.19	1.87	2.10
400	14.0	2.96	2.50	2.79
500	18.6	3.77	3.13	3.48
600	23.0	4.61	3.75	4.15
700	27.3	5.47	4.36	4.82
800	31.7	6.29	4.95	5.45
900	35.9	7.04	5.47	6.07
1000	40.0	7.67	5.90	6.58

have been made so as to provide data for relative line intensities if the sensitivity of the galvanometer had been measured as used for each line, but since a different method, described in the next section, was considered to be better, it was thought inadvisable to interfere with the electrical circuit of the photo-cell and galvanometer during this series of readings.

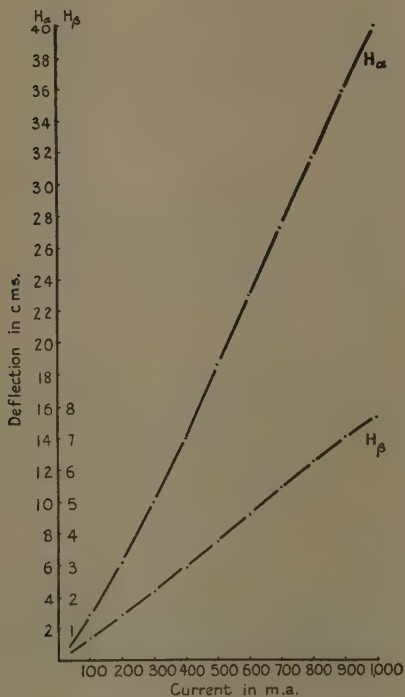
Section II.

Variation of Relative Intensities of the Lines with Increasing Current.

The method adopted was to measure the deflexion produced by the four lines separately, when the current was steady at one particular value, then to alter the current

and leave the apparatus to come to a steady state at a different current and measure the four deflexions again. This procedure was repeated as the current was altered, in steps, over the range 190 m.a. to 970 m.a., the pressure being maintained steady at 0.18 mm. of Hg, whilst the measurements were being made, and curves 5 and 6

Fig. 5.

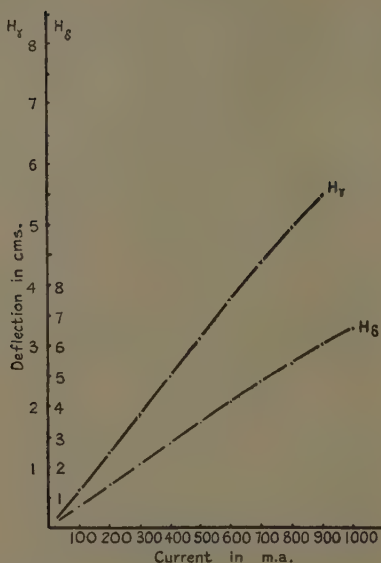


Variation of the intensities of H_α and H_β at a constant pressure of 0.38 mm.

were used to obtain the small corrections needed to allow for slight variations of the current during the measurements of the four lines.

Table II. and fig. 7 give the results of the measurements, the ratios of the deflexions having been converted into ratios of intensities.

Fig. 6.



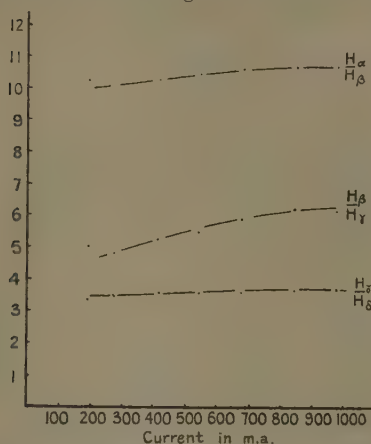
Variation of the intensities H_γ and H_δ at a constant pressure of 0.38 mm.

TABLE II.

Variation of Relative Intensities with Increasing Current at a Constant Pressure of 0.18 mm. of Hg.

Current in m.a.	H_α/H_β .	H_β/H_γ .	H_γ/H_δ .
190	10.25	5.04	3.34
270	10.10	4.84	3.49
410	10.30	5.30	3.57
540	10.50	5.46	3.56
675	10.65	5.90	3.53
840	10.75	6.24	3.61
970	10.75	6.14	3.69

Fig 7.



The variation of intensities of the lines at a constant pressure of 0.18 mm. of Hg.

TABLE III.

Intensity of each Line and Relative Intensities of Lines with varying Pressure at a Constant Current of 800 m.a.

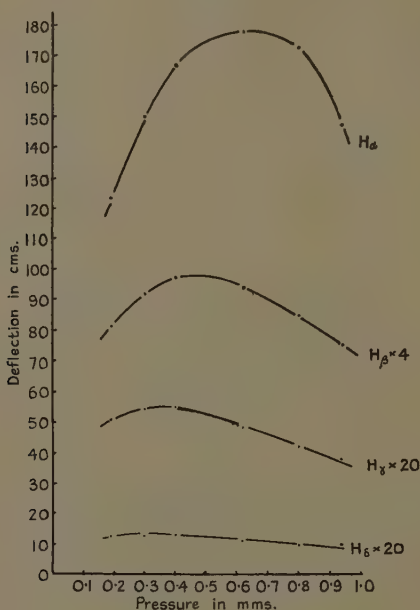
Pressure in mm.	Deflexions in centimetres.				Relative Intensities.		
	H_{α} .	H_{β} .	H_{γ} .	H_{δ} .	H_{α}/H_{β} .	H_{β}/H_{γ} .	H_{γ}/H_{δ} .
0.19	123.4	20.30	2.55	0.62	11.0	6.22	3.52
0.30	150.0	22.90	2.71	0.62	11.8	6.62	3.74
0.40	167.0	24.30	2.75	0.65	12.4	6.93	3.62
0.62	177.8	23.50	2.40	0.54	13.6	7.67	3.78
0.80	172.5	21.20	2.08	0.48	14.7	7.98	3.71
0.94	147.2	18.80	1.90	0.50	14.1	7.75	3.25

Section III.

The galvanometer was kept at the same sensitivity throughout the series of measurements in this section, and the current was kept as steady as possible at 800 m.a. The pressure was altered, in steps, from 0.19 mm. to 0.94 mm., and the deflexions for H_{α} , H_{β} , H_{γ} , and H_{δ} were measured at each pressure. Table III. gives the deflexions

produced by each line and also the relative intensities, the variations being shown in figs. 8 and 9.

Fig. 8.



The variation of intensity of the lines separately at a constant current of 800 m.a.

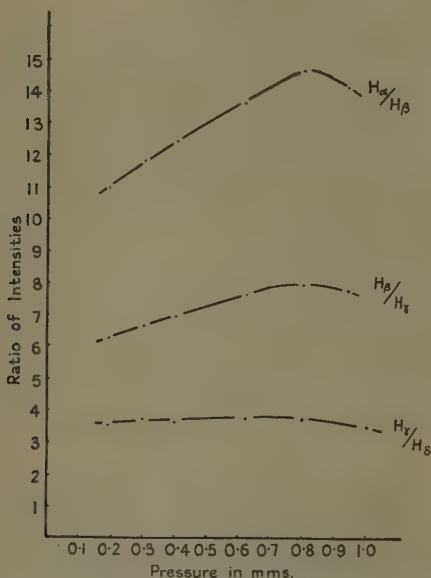
Two important points are to be noticed :—

- (a) The intensity of each line comes to a maximum and then decreases.
- (b) The abscissa of the maximum decreases in the order H α , H β , H γ , H δ .

The measurements in this section and those in Section II. are entirely independent, apart from the calibration of the photo-cell, and allow of a measure of the consistency of the results being made. The results in Section II. give, for a current of 800 m.a. and a pressure of 0.18 mm., $H_\alpha/H_\beta=10.7$, $H_\beta/H_\gamma=6.1$, $H_\gamma/H_\delta=3.6$, whilst for the

same conditions the respective values of the ratios are in this Section 10.9, 6.2, 3.6.

Fig. 9.



Variation of ratio of intensities of lines with pressure at a constant current of 800 m.a.

Interpretation and Discussion of Results.

It is seen that the relative intensity of H_α to H_β differs greatly from the value 2.18 given by quantum mechanics for an isolated hydrogen atom. In this connexion, as pointed out by Ornstein and Lindeman *, it is interesting to note the remark of Elsasser that the ratio of the excitation probability for H_α to that for H_β will be equal to the absorption ratio for these lines. Quantum mechanics gives 5.37 as the value of this ratio, and this has been verified experimentally by Snoek and Ornstein †, and therefore the calculated value of the relative intensities of

* *Zeits. f. Physik*, lxiii. p. 18 (1930).

† *Zeits. f. Physik*, l. p. 600 (1928).

H_α and H_β should be multiplied by 5.37. Thus H_α/H_β
(calculated)

$$= 2.18 \times 5.37$$

$$= 11.7$$

*The Number of Creations in a given State, as a
Function of the Current.*

Suppose we consider the change of intensity of a line as the current changes, the other factors remaining constant, an excited atom may change its state as a result of a transition, which may be spontaneous or stimulated by radiation or be brought about by collisions with an atom, ion, or electron. The density of the radiation is small and the contribution to the observed intensity by stimulated transitions can be neglected. With regard to the effect of collisions, the number of collisions of any one atom with all others depends on the temperature and the pressure. If we consider the atoms which emit the Balmer line H_α , then if

N_3 = number of atoms/c.c. in the 3 quantum state
and

ν = total number of atoms/c.c.

which is inversely proportional to T , then the number of collisions per sec. per c.c. which involve the N_3 atoms will depend on the product of N_3 , ν , $T^{\frac{1}{2}}$, and σ^2 , where T is the absolute temperature corresponding to the kinetic energy of the atoms and σ is an average collision diameter for atoms in the 3 state, and thus the number of collisions involving atoms in the 3 state will depend on $N_3 T^{-\frac{1}{2}}$. Of these collisions suppose a fraction α_3 result in a transition in a second, so that the number leaving the 3 state per sec. per c.c. will be $N_3 \alpha_3 T^{-\frac{1}{2}}$ on account of collisions. By spontaneous transition a number $N_3(A_{32} + A_{31})$ leave the 3 state per sec./c.c., so that we may write, number of atoms leaving the 3 state per c.c./sec. = $N_3 A_3 + \alpha_3 N_3 T^{-\frac{1}{2}}$, where

$$A_3 = \sum_{n=1}^{n=2} A_{3n}.$$

Now the voltage across the tube remains approximately constant, so that the drift velocity of the ions will remain nearly constant at different currents, so that in order to carry the current the number of atoms/c.c. which are

ionized may reasonably be written as proportional to C the current. Also the ratios H_α/H_β , H_β/H_γ , etc. determine the ratios N_3/N_4 , N_4/N_5 , etc. These ratios certainly increase, but the rate of increase is small, so that over a large variation of current, at constant pressure, the distribution of the atoms over the quantum states is roughly constant, although the population density of the lower states tends to increase as the discharge current increases.

Let the number of atoms created per c.c. per sec. in the n th state $=f_n(c)$ where "created" refers to atoms arriving at the n th state by any means except that of transition from a higher state.

Then the number entering the n th state

$$=f_n(c)+\sum_{r=1}^{r=\infty} N_{n+r} A_{n+r, n}$$

and the number leaving the n th state

$$=N_n A_n + \alpha_n N_n T^{-\frac{1}{2}}.$$

Suppose we consider in detail the case of the state 3. We have for a steady state

$$f_3(c)+\sum_{r=1}^{r=\infty} N_{3+r} A_{3+r, 3} = N_3 A_3 + \alpha_3 N_3 T^{-\frac{1}{2}}.$$

Now if α_3 = probability of one collision causing a transition from the 3 state, and assuming the effective collision diameter of a H atom in the 3 state is 10^{-7} cm. and the temperature about 600° A., we can calculate the number of collisions and obtain the relation $\alpha_3 \sim 10^9 \alpha_3$.

$$\begin{aligned} \text{Also } \sum_{r=1}^{r=\infty} N_{3+r} A_{3+r, 3} &= N_4 A_{43} + N_5 A_{53} + \dots \\ &= \frac{P_\alpha}{h\nu_{43}} + \frac{P_\beta}{h\nu_{53}} + \dots \\ &= \frac{1}{h\nu_{43}} \left[P_\alpha + P_\beta \frac{\nu_{43}}{\nu_{53}} + \dots \right]. \end{aligned}$$

$$\therefore f_3(c) + \frac{1}{h\nu_{43}} [P_\alpha + \dots] = N_3 A_3 + \alpha_3 N_3 T^{-\frac{1}{2}}.$$

$$\text{Writing } \Sigma \text{ for } \left[P_\alpha + P_\beta \frac{\nu_{43}}{\nu_{53}} + \dots \right]$$

$$\text{we get } \frac{h\nu_{32} f_3(c)}{\Sigma} + \frac{\nu_{32}}{\nu_{43}} = \frac{1}{A_{32}} \frac{H_\alpha}{\Sigma} [A_3 + \alpha_3 T^{-\frac{1}{2}}]$$

a simple reduction leads to

$$f_3(c) + \frac{20}{7} k N_3 A_{32} = N_3 [A_3 + \alpha_3 T^{-\frac{1}{2}}], \quad \text{where } k = \frac{\Sigma}{H_\alpha}.$$

$$\text{Hence} \quad H_\alpha = \frac{f_3(c) h \nu_{32} A_{32}}{[A_3 + \alpha_3 T^{-\frac{1}{2}}] - \frac{20k}{7}}.$$

To get the value of k we may proceed as follows :—

$$\frac{P_\alpha}{H_\beta} = \frac{\nu_{43}}{\nu_{42}} \cdot \frac{A_{43}}{A_{42}}; \quad \frac{P_\beta}{H_\beta} = \frac{\nu_{53}}{\nu_{52}} \cdot \frac{A_{53}}{A_{52}} \cdot \frac{H_\gamma}{H_\beta}; \text{ etc.}$$

$$\therefore \quad \frac{1}{H_\beta} \left[P_\alpha + \frac{\nu_{43}}{\nu_{53}} P_\beta + \dots \right] = 0.356,$$

using Slack's values of $A_{n\alpha}$, and the values of $\frac{H_\alpha}{H_\beta}$, $\frac{H_\beta}{H_\gamma}$, etc. for the current 540 m.a.

$$\therefore \quad \frac{\Sigma}{H_\alpha} = 0.034 = k.$$

$$\begin{aligned} \therefore \quad H_\alpha &= \frac{f_3(c) h \nu_{32}}{2.16 + 22.6 a_3 T^{-\frac{1}{2}}}, \\ &\sim \frac{f_3(c) h \nu_{32}}{(2.16 + a_3)}. \end{aligned}$$

Now a_3 is probably very small, and since H_α is practically a linear function of the current we have also $f_3(c) = Kc$, and any departure from a straight line relationship between the intensity of a line as shown for H_β , H_γ , etc. and the current is probably due to the falling off of the number of creations in the higher states at large currents. It is shown in a later paragraph that if we assume thermodynamic equilibrium, then the T which gives the determined ratio N_3/N_4 decreases with increasing current. It is difficult to attach a meaning to T , and it is possible that measurements of the visibility of a system of interference fringes using one of the Balmer wave-lengths, along the lines of the experiments of Fabry and Buisson, might give some idea of the temperature. It is almost certain that the result of such measurements would show an increase in temperature with increase in current, in effect proving that thermodynamic equilibrium is not established in a discharge-tube.

The Effect of Change of Pressure.

The results obtained when the intensities were measured at varying pressures, but at constant discharge current, are very striking, the graphs rising to a maximum and falling away again. This can be explained by attributing the increasing intensity to the increasing number per c.c. of atoms whose state can be raised. This differs from the preceding paragraph in that the pressure now is being altered and the transfer of charge depends on the mean free-path as well as the number of ions, and at higher pressures the mean free-path is less, and, if the effective temperature remains approximately constant, the number of ions per c.c. must increase, even if the current remains the same. The increase in the number of ions will be approximately proportional to p^2 , whilst the number of collisions of any one atom will be proportional to p , and therefore as p increases the number of collisions increases more rapidly than does the number of ions.

It has been known for some time that, in order to produce the higher members of the Balmer series, it was necessary to use a low pressure in the discharge-tube. It is seen from the graphs that the effect of increasing pressure is apparent at a smaller pressure the higher the member of the Balmer series under investigation, and thus the highest lines become quenched first with increasing pressure. It is easily noticed that the graphs would not pass through the origin if they were reasonably extended towards zero pressure, but this is not surprising in view of the condition that the values are for a discharge current of 800 m.a., and the voltage and conditions in the tube would be altered to a great extent if a current of 800 m.a. was maintained as the pressure approached zero. Accordingly, therefore, it is not in keeping with the conditions imposed to extrapolate to zero pressure.

The Number of Creations in a given State as a Function of the Pressure.

Let $f_n(p)$ = No. created/c.c./sec. in quantum state n at the pressure p mm. of mercury.

N_n = No./c.c. in state n at any time.

A treatment similar to the case for the function $f_3(c)$ gives the result

$$H_x = \frac{f_3(p)}{2.16 + 8a_3p} \quad (p \text{ in mm. of Hg.})$$

The numerical factor multiplying a_3p in the denominator increases for higher lines and may possibly have an effect on the intensities of the high members, but for the lower members the form of $f(p)$ almost certainly governs the form of the intensity-pressure relation.

Effective Temperatures.

By making use of the Einstein A coefficients, as computed by Slack*, the relative numbers of atoms in different quantum states can be found, and it is interesting to find a value of the absolute temperature which would give the calculated value of N_3/N_4 , say, if we assume that the atoms are in thermodynamic equilibrium.

If g_s is the statistical weight of the s th quantum state then if $T_{s_1s_2}$ is the value of the "Effective Temperature" in °K. as calculated from the ratio N_{s_1}/N_{s_2} , then we have

$$\frac{N_{s_1}}{N_{s_2}} = \frac{g_{s_1}}{g_{s_2}} e^{\frac{h\nu_{s_2s_1}}{kT_{s_1s_2}}},$$

or, for purposes of calculation,

$$\log_{10} \left(\frac{N_{s_1}}{N_{s_2}} \cdot \frac{g_{s_2}}{g_{s_1}} \right) = \frac{6210}{\lambda_\mu T_{s_1s_2}},$$

where λ_μ is the wave-length in μ of the line emitted by the transition $s_2 \rightarrow s_1$, and so, knowing the ratio $N_{s_1} : N_{s_2}$ the value of $T_{s_1s_2}$ can be calculated. Taking the results showing the variation of the relative intensities with current and calculating the effective temperature for each value of the ratios we have the results collected in Table V.

It is easily seen from the expression for $T_{s_1s_2}$, and the fact that N_3/N_4 , etc. increases with increasing current, that the "effective temperature" decreases with increasing current, and it is clear from the table that no single temperature will account for all the ratios N_3/N_4 , N_3/N_5 , N_3/N_6 †.

* Slack, Phys. Rev. xxxi. p. 527 (1928).

† See also Ornstein and Lindemann, *Zeits. f. Physik*, lxiii. p. 19 (1930).

Number of Dispersion Electrons.

In the determination of the relative number of dispersion electrons corresponding to H_α and H_β , either by measurements of anomalous dispersion * or by absorption coefficient measurements †, there arises a factor of the form

$$1 - \frac{N_{\kappa_1}}{N_j} \cdot \frac{g_j}{g_{\kappa_1}},$$

$$1 - \frac{N_{\kappa_2}}{N_j} \cdot \frac{g_j}{g_{\kappa_2}},$$

which is introduced to correct for "negative dispersion" in one case and "negative absorption" in the other

TABLE V.
Showing "Effective Temperatures."

Press. in mm.	Current in m.a.	N_3/N_4	N_3/N_5	N_3/N_6	T_{34}	T_{35}	T_{36}
0.18	970	2.78	5.90	9.03	4780	3970	3650
"	840	2.77	5.74	8.77	4800	4010	3690
"	675	2.75	5.45	8.34	4820	4090	3740
"	540	2.71	5.10	7.75	4870	4190	3820
"	410	2.66	4.71	7.02	4910	4320	3930
"	270	2.60	4.38	6.40	4990	4430	4040

case; these, of course, being different aspects of the same effect. In the theory of dispersion $f_{\kappa j}$ is written for the oscillator strength

$$A_{\kappa j} \frac{g_{\kappa}}{g_j} \cdot \frac{mc^3}{8\pi^2 e^2 \nu_{\kappa j}^2},$$

and if we write $\eta_{\kappa j}$ for the number of dispersion electrons associated with the transition between the quantum states k and j , then

$$\eta_{\kappa j} = N_j f_{\kappa j} \left(1 - \frac{N_{\kappa}}{N_j} \cdot \frac{g_j}{g_{\kappa}} \right),$$

* Ladenburg, *Zeits. f. Physik*, xlviii. pp. 15-16 (1928); Carst und Ladenburg, *Zeits. f. Physik*, l. (1928).

† Snoek and Ornstein, *Zeits. f. Physik*, l. (1928).

and we have

$$\frac{f_{H_\alpha}}{f_{H_\beta}} = \frac{\eta_\alpha}{\eta_\beta} \cdot \frac{1 - \frac{N_4}{N_2} \cdot \frac{g_2}{g_4}}{1 - \frac{N_3}{N_2} \cdot \frac{g_2}{g_3}}.$$

From the results in this paper, to take a specific case, we have at 0.18 mm. pressure and at a current of 675 m.a.

$$N_3 : N_4 : N_5 : N_6 = 83.3 : 30.3 : 15.3 : 10$$

and an extrapolation of these values was made; although the accuracy of the value of N_2 obtained is not great, the value of the correcting factor given above is not greatly affected. For the values of the pressure and current just quoted the relative values of N_n are

$$N_2 : N_3 : N_4 : N_5 : N_6 = 540 : 83.3 : 30.3 : 15.3 : 10.$$

Now the theoretical value of $\frac{f_{H_\alpha}}{f_{H_\beta}} = 5.37$, and using this and the correcting factor calculated from the above ratios, we have

$$5.37 = \frac{\eta_\alpha}{\eta_\beta} 1.060,$$

$$\therefore \frac{\eta_\alpha}{\eta_\beta} = 5.07.$$

As a result of their measurements, Carst and Ladenburg give the experimental value of $\frac{\eta_\alpha}{\eta_\beta}$ as 4.66 as an average value of results taken over currents varying from 200 m.a. to 400 m.a. and pressures varying from 0.3 mm. to 0.9 mm.

Extension of Calculations of Ratio H_α/H_β .

Cillié* has derived an expression for the number of captures on the various hydrogen levels, and his Table I. shows that faster electrons, *i. e.*, those with energies corresponding to higher temperatures, tend to be caught more often on the lower levels. The results of Section II. show that the ratio of the number of atoms in a lower state to the number in a higher state increases with increasing current, and connecting this with Cillié's

* M.N.R.A.S. xcii. p. 820 (1932).

calculations suggests that the temperature increases with increasing current.

Table III. in Cillié's paper, giving the Balmer decrement, does not take into account captures on levels higher than the 7th and gives the decrement for three temperatures. I have extended this table, taking into account the captures on the levels higher than the 7th, assuming that the effective coefficient for terms higher than M(7T) is 1.035 for equation (α) and 1.860 for equation (β). In the calculations for the temperature 1000°, A/n^2 is not small, and the expression for S was extended, giving

$$S < 2e^{A/121} \sum_{n=11}^{n=\infty} \frac{1}{n^3} \log n + A \sum_{n=11}^{n=\infty} \frac{1}{n^5} e^{A/n^2} \\ - (\gamma + \log A) \sum_{n=11}^{n=\infty} \frac{1}{n^3} e^{A/n^2} - \frac{1}{4} \sum_{n=11}^{n=\infty} \frac{1}{n^7} A^2 e^{A/n^2} \\ + \frac{A}{18} \sum_{n=11}^{n=\infty} \frac{1}{n^9} e^{A/n^2}.$$

Thus, taking into account neglected captures and other terms in the expression for S in the case where A/n^2 was not small, Table III. of Cillié's paper becomes, for H_α and H_β ,

	T=1000°.	T=5000°.	T=10,000°.	T=20,000°.	T=50,000°.
$H_\alpha \dots$	22.0	24.8	25.5	27.0	28.7
$H_\beta \dots$	10	10	10	10	10

This table is an extension of his Table II. and still shows $N_4 > N_3$, and therefore does not describe the conditions prevalent in the hydrogen discharge-tube where $N_5 < N_4 < N_3$.

Summary.

Examination by a cathode-ray oscillograph of the characteristics of a discharge-tube excited by unsmoothed supply shows the presence of high kicks of P.D. for small maximum currents at the frequency used, and it is shown that these can be eliminated even with unsmoothed supply by increasing the frequency, a point which may be of electrotechnical interest. Then, using a well-smoothed D.C. supply, a detailed investigation of the

effect of pressure and discharge current on the intensities, and on the relative intensities, of the Balmer lines has been carried out. A discussion of the results leads to the conclusions (a) that thermodynamic equilibrium is not established in a discharge-tube, (b) the excitation-function governs the variation of the intensities of the lower members of the Balmer series, (c) the ratio of the number of dispersion electrons associated with H_α and H_β is 5.07 at 0.18 mm. and 675 m.a., (d) Cillié's treatment of the Balmer decrement does not give results in accord with experiment. An experiment is suggested whereby information regarding the temperature of the gas in the discharge-tube may be obtained.

In conclusion I wish to express my indebtedness to Professor J. A. Carroll, whose interest and advice throughout the course of the work have been most helpful.

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LI. *Studies on the Oxidation of Metals.*—Part IV. *The Oxide Film on Aluminium.* By G. D. PRESTON, M.A., and L. L. BIRCUMSHAW, M.A., D.Sc.* (From the National Physical Laboratory.)

[Plate IV.]

THE nature of the oxide film which confers on aluminium and on many alloys containing it in small percentages such remarkable power to resist corrosion is a matter of considerable technical importance. It has been known for some years that the oxide film on molten aluminium is a crystalline film of the cubic form (γ) of Al_2O_3 , and is not the stable hexagonal (α) corundum. A note to this effect was published ⁽¹⁾ by the present authors and the result confirmed by Steinheil ⁽²⁾, and subsequently by other investigators. This fact was discovered by transmitting a beam of electrons through films of oxide prepared by heating aluminium foil mounted on a wire loop until the metal fused. On cooling, a net-

* Communicated by Prof. C. H. Desch, D.Sc., Ph.D., F.R.S., F.I.C.

work of metal supports the oxide film. Specimens were also obtained by withdrawing a loop of wire from molten aluminium, a film of the oxide adhering to the loop. In all cases the electron diffraction pattern could be ascribed to $\gamma\text{Al}_2\text{O}_3$. In spite of the ease with which patterns of this form of the oxide could be obtained by the transmission method, no indication of the presence of a film having this structure could be found when specimens of aluminium were examined by the reflexion method, the surface being polished and oxidized at room temperature or at temperatures up to about 400°C . The surface of cold-rolled aluminium yielded a diffraction pattern consisting of two broad rings, for which the values of $\sin \theta/\lambda$ were found to be 0.21 and 0.375 respectively. These two figures are in the ratio of 1:0.57, and it seemed, for this reason, that the pattern arose from the highly cold-worked surface. Raether⁽³⁾, and Darbyshire and Dixit⁽⁴⁾ have shown that certain metals in the cold worked state produce a diffraction pattern of this type, and have attributed the pattern to the existence on the surface of a layer of amorphous material. If we assume that the scattering is due to pairs of atoms separated by a distance D , then (following Ehrenfest⁽⁵⁾, and Zernicke and Prins⁽⁶⁾) the intensity maxima will occur

near the values of kD which make $\frac{\sin kD}{kD}$ a maximum,

when $k=4\pi \sin \theta/\lambda$. The appropriate values of kD are 7.72 and 14.03 for the two haloes, giving $D=2.92\text{ \AA}$. and $2.97\text{ \AA}$., respectively, from the values of $\sin \theta/\lambda$ observed. The value is a little larger than the distance $2.85\text{ \AA}$. between aluminium atoms in the metal.

When a specimen of cold-rolled aluminium which showed these two haloes on its electron diffraction diagram was heated for 1.5 hours at 450°C . (a treatment which should remove all traces of cold work), the electron diffraction diagram still consisted of two haloes. Whether these are identical with the pairs observed in the material before heat treatment is not certain. They are at any rate so similar that it appeared, in the case of aluminium, that the diffraction effect was not due to amorphous metal but to an amorphous oxide layer on the surface.

In view of the failure to observe diffraction effects from the oxide film, which is commonly believed to be

present on solid aluminium (unless the haloes observed in the spectrum of the cold-rolled metal are in reality due to an oxide film), it was decided to continue the investigation by examining specimens of film separated from the metal by treatment in HCl gas. This method has been used by Withey and Millar⁽⁷⁾ to isolate the pellicle of oxide on massive specimens of aluminium, and we have found it possible to remove the metal from foil in the same way, leaving the oxide in the form of a thin sheet.

Preparation of the Films.

The thin film of oxide on the surface of aluminium foil was separated from the metal by exposing the foil to dry HCl gas at about 200–250° C. The gas was generated by the action of concentrated sulphuric acid on fused sodium chloride, and was dried by passing through two sulphuric acid tubes and two tubes packed with asbestos and phosphorus pentoxide. The foil (6×3 cm.), rolled into a cylinder about 1 cm. diameter, lay in a small silica boat in the middle of a silica tube 50 cm. long, supported over a bunsen burner. After passing through this tube the gas entered a drying tube packed with silica gel, a bottle containing solid potash, and finally a mercury seal to prevent air being sucked back into the system, and to maintain a small pressure of a few millimetres of mercury in excess of atmospheric pressure in the whole system.

At the beginning of an experiment the whole apparatus, including the HCl generator, was swept out by a stream of pure dry hydrogen for at least one hour. Sulphuric acid was then admitted to the flask containing the fused sodium chloride, and the HCl was allowed to pass over the specimen at room temperature until it could be seen to be reacting with the potash. The silica tube in the neighbourhood of the foil was then heated slowly by a gas flame to about 200° C. The start of the reaction of the HCl with the foil could be seen by the condensation of a layer of aluminium chloride in the cold part of the tube. The reaction always took some time to start, but, once started, it proceeded quite rapidly. The whole process usually occupied between one and two hours from the time at which heating was started. The removal of the metal left a thin film in the boat, which

retained the shape of the cylinder into which the foil had been rolled.

Thickness of the Film.

The films obtained by the above method were quite transparent, occasionally with a nacreous lustre. A number of them obtained from the same area of foil were weighed immediately after preparation and after ignition for one hour in air at about 900° C. with the following results :—

As prepared.	Ignited for 1 hour at 900° C.
mg.	mg.
0.203	0.169
0.165	0.083
0.193	0.101
—	0.194
—	0.092
—	0.167

The considerable loss of weight on ignition may be due to the removal of water-vapour absorbed during removal of the film from the tube in which it was prepared. As shown by the electron diffraction photograph, the film as prepared is practically amorphous, while ignition at 900° C. produces crystallization. The large variation in weight may be due either to failure to remove the last traces of aluminium chloride in the preparation of the specimens or to variations in the real area of the foil due to surface irregularities. The real area may be about double the geometrical area of the piece of foil. Part, or even all, of the variations of weight may be due to differences in the amount of HCl adsorbed by the film during preparation. The films were cooled in a stream of dry HCl, and were no doubt saturated with gas when removed from the tube in which they were prepared. We are greatly indebted to Miss I. Hadfield, M.Sc., for carefully carrying out the weighings recorded above.

The thickness of the film can be approximately estimated from these data. The maximum weight of the film is about 2×10^{-4} gm. It covers both sides of a

piece of foil 18 sq. cm. in area, so that, assuming a density equal to that of γ Al_2O_3 , i. e., 3.6 gm./ml., the thickness is not greater than $2 \times 10^{-4} / 36 \times 3.6 = 1.5 \times 10^{-6}$ cm. In some cases the weight is about half, and the real area may be double that assumed in this calculation, in which case the thickness would be one-fourth of the above figure.

Results of Electron Diffraction Examination.

Pieces of the film were mounted, without the use of any adhesive, on small loops of platinum wire so that the beam of electrons could be transmitted through them. The diffraction photographs obtained from the film as prepared consisted of two broad haloes, for which the values of $\sin \theta / \lambda$ were found to be approximately 0.17 and 0.37. On some of the photographs, which had been over-exposed, a few faint, rather broad rings appeared, and the inner halo became a rather intense disk of blackening. The outer of these two haloes is coincident with the outer of the two observed by reflexion from cold-rolled aluminium.

The effect of heating the film in air at different temperatures was next investigated. As it had been exposed to a temperature in the neighbourhood of 250° C. in the process of preparation, it was important to ascertain whether this temperature had produced any change in the film. We found that exposure to temperatures as high as 470° for periods longer than that occupied by the preparation of the film did not produce any change in the electron diffraction photographs. One specimen, which was heated at 470° C. for 1.5 hours and then at 620° C. for a further 1.5 hours, showed no change. A further treatment at 790° C. for 2 hours produced the ring pattern typical of γ Al_2O_3 , the oxide known to be present on the surface of molten aluminium at that temperature. To ascertain whether crystallization would take place at a temperature just above the melting point of aluminium another specimen was subjected to the following treatment :—

10 hours at	690° C.
10 " "	680° C.
50 " "	680° C.
2 " "	800° C.
1 hour "	1000° C.

Electron diffraction photographs were taken of the specimen as prepared and after each heat treatment (Pl. IV. figs. 1-6). The photographs show that after the treatment a change has taken place in the structure of film, a rather intense ring of small diameter making its appearance (Pl. IV. fig. 2). The subsequent treatments at 680° C. improved the sharpness of this ring and others appeared (Pl. IV. fig. 3), some of which have the spacings characteristic of $\gamma\text{Al}_2\text{O}_3$. After the second treatment at 680° C. the most intense rings correspond very exactly with those obtained by heating the film at the highest temperature, 1000° C. This last photograph (Pl. IV. fig. 6) has been ascribed to $\gamma\text{Al}_2\text{O}_3$, and a rather weak inner ring is present, but its diameter is not the same as that observed in the photographs obtained from the specimen at the lower temperature. In the $\gamma\text{Al}_2\text{O}_3$ photographs the weak inner ring corresponds to reflexion from a plane of indices (111). (According to Verwey⁽⁸⁾ the X-ray photographs of $\gamma\text{Al}_2\text{O}_3$ show no (111) ring.) In the photographs of the film treated at the lower temperatures the diameter of the inner ring is too large to be ascribed to reflexion by (111) planes and too small to arise from (200) planes. This fact will be made clear by the figures given in Table I. The diameters of the most intense rings on the photographs were measured by a travelling microscope and, judging by the way in which readings could be repeated, are correct to ± 0.1 mm. In the table the values of D , the ring diameters, are recorded together with an estimate of the intensity of the ring. The values of $D/\sqrt{h^2+k^2+l^2}$ are also given. This quantity should be a constant for each photograph equal to $2\lambda L/a$, where λ is the wave-length, L the distance between specimen and photographic plate (34.5 cm.), and a the side of the unit cube of $\gamma\text{Al}_2\text{O}_3$.

The figures given in the last two columns, referring to two photographs taken with different voltages, show that in the case of the material after the final heat treatment at 1000° C. the inner ring agrees well with the assumption that it arises from the planes (111). The mean of the value of $D/\sqrt{h^2+k^2+l^2}$ for the last five rings when multiplied by $\sqrt{3}$ is nearly equal to the observed diameter of the ring, *e.g.*, $4.62 \times \sqrt{3} = 8.0$ mm., the observed figure being 8.1 mm. In the other four columns

TABLE I.

$hkl. \ h^2+k^2+l^2.$		70 hours at 680° C.				
		$\lambda=0.052 \text{ \AA}.$		$\lambda=0.075 \text{ \AA}.$		
		D (mm.). $D/\sqrt{h^2+k^2+l^2}.$		D (mm.). $D/\sqrt{h^2+k^2+l^2}.$		
111	3	s	8.75	{ 5.05 }	m 12.6	{ 7.28 }
200	4		—	{ 4.37 }		{ 6.30 }
220	8		—	—	w 18.8	6.65
311	11	w	14.9	4.49	mw 21.9	6.60
400	16	w	18.25	4.56	w 26.4	6.60
333	27		—	—	—	—
511			—	—		—
440	32	w	26.0	4.60	w 37.35	6.60
Mean, excluding			4.55 ± 0.04		6.61 ± 0.02	
(111) & (200).}			7.89		7.83	
$\alpha=$						

		2 hours at 800° C.				
		$\lambda=0.052 \text{ \AA}.$		$\lambda=0.052 \text{ \AA}.$		
111	3	s	8.55	{ 4.93 }	vs 8.65	{ 5.00 }
200	4		—	{ 4.28 }		{ 4.33 }
220	8		w 12.7	4.49	—	—
311	11	vs	14.8	4.46	s 14.90	4.49
400	16	s	17.95	4.49	s 18.05	4.51
333	27	w	23.2	4.46	—	—
511			—	—		—
440	32	s	25.35	4.48	s 25.5	4.51
Mean, excluding			4.47 ± 0.1		4.50 ± 0.01	
(111) & (200).}			8.03		7.97	
$\alpha=$						

		1 hour at 1000° C.				
		$\lambda=0.052 \text{ \AA}.$		$\lambda=0.075 \text{ \AA}.$		
111	3	w	8.1	4.63	w 11.5	6.64
200	4		—	—		—
220	8	s	13.1	4.63	s 18.6	6.58
311	11	vs	15.3	4.61	vs 21.7	6.55
400	16	s	18.5	4.62	s 26.25	6.57
333	27	w	24.1	4.64	w 34.1	6.57
511			—	—		—
440	32	s	26.1	4.61	s 37.05	6.55
Mean, excluding			4.62 ± 0.02		6.56 ± 0.01	
(111) & (200).}			7.77		7.89	
$\alpha=$						

this is not the case; we have, in fact, the following comparison:—

Column 1 ..	$4.55 \times \sqrt{3} = 7.9$ mm.	observed	8.75 mm.	$d = 4.11$ Å.
Column 2 ..	$6.61 \times \sqrt{3} = 11.45$ „	„	12.6 „	$d = 4.13$ „
Column 3 ..	$4.47 \times \sqrt{3} = 7.75$ „	„	8.55 „	$d = 4.21$ „
Column 4 ..	$4.50 \times \sqrt{3} = 7.80$ „	„	8.65 „	$d = 4.16$ „

The difference amounts to much more than the error of measuring the ring diameters. Moreover, to obtain the observed figures for the ring diameters the factor $\sqrt{3}$ would have to be replaced by $\sqrt{3.7}$, so that the ring cannot be ascribed to any plane in the cubic structure of $\gamma\text{Al}_2\text{O}_3$. The mean value of the parameter of $\gamma\text{Al}_2\text{O}_3$ from these six photographs is 7.90 ± 0.07 Å., in agreement with the X-ray figure.

It is clear, therefore, that the intense inner ring which makes its appearance on the first heating at 680°C . is not due to $\gamma\text{Al}_2\text{O}_3$. The spacings observed from a photograph of the film after 20 hours at 680°C . are given in Table II., together with a list of the rings observed in the photographs of $\gamma\text{Al}_2\text{O}_3$ and the spacings calculated for $a = 7.90$ Å. It will be seen that with the exception of the first ring most of the lines are in fair agreement with the assumption that the crystalline phase is $\gamma\text{Al}_2\text{O}_3$. The accuracy with which the rings can be measured in the photographs of the film in the early stages of crystallization is not high, so that some error is unavoidable. Those rings to which the letter γ has been added would seem to be ascribable to $\gamma\text{Al}_2\text{O}_3$, but there is no doubt that the inner intense ring must be explained in some other way. Steinheil⁽²⁾ has reported the existence of a form of alumina which he calls $\epsilon\text{Al}_2\text{O}_3$, to which he assigns a face-centred cubic structure of side 5.35 Å. This structure, however, cannot produce a ring of spacing 4.09 Å., and none of the rings recorded in Table II. are characteristic of this phase. Nor can it be ascribed to yet another form of Al_2O_3 described by Verwey⁽⁹⁾.

The photographs obtained from the film as prepared are of some interest, and bear a marked resemblance to the X-ray diffraction photographs yielded by films of electrolytically oxidized aluminium foils which have been investigated by Schmid and Wassermann⁽¹⁰⁾. These authors show that the oxide film (technically

known as Eloxal) gives an X-ray photograph consisting of two broad rings, the inner one merging into the blackening produced by the incident beam. The outer ring has the same diameter as the most intense ring in the X-ray photographs of γ Al_2O_3 , into which the Eloxal film is transformed by annealing for 48 hours at 600°C . The electron diffraction photographs obtained in the present work have been analysed to see whether the superficial resemblance to the X-ray photographs of Eloxal can be substantiated. For this purpose micro-

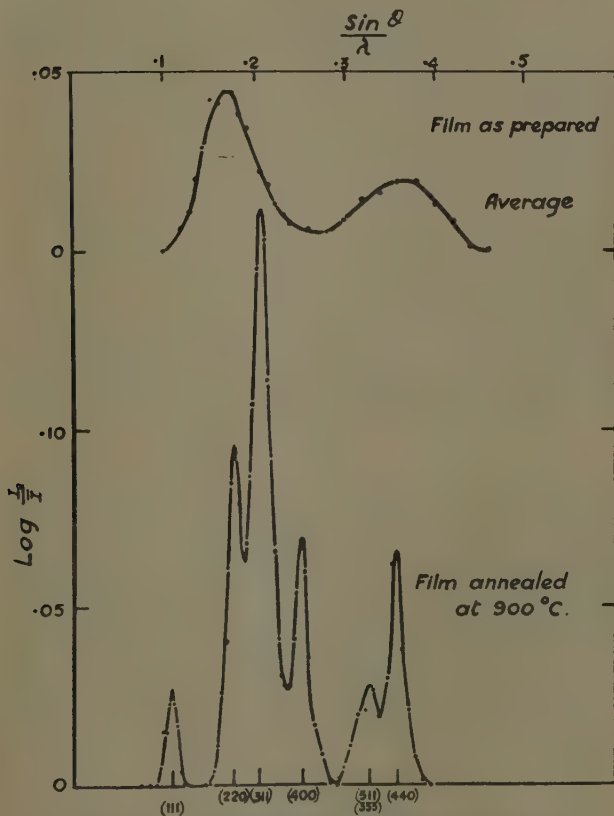
TABLE II.
20 hours at 680°C .

I.	d (Å.).	I.	γ Al_2O_3 .	
			d .	hkl .
vs	4.09	mw	4.55	111
vw	3.27			
vw	2.77 γ	s	2.79	220
s	2.47	vs	2.38	311
s	2.00 γ	s	1.97	400
w	1.65 γ	w	1.61	422
w	1.51 γ	m	1.52	{ 511
				{ 333
s	1.40 ₅ γ	s	1.39 ₅	440
w	1.20 γ	mw	1.19	622
vw	1.06 γ	mw	1.14	444
vw	1.00	w	1.03	{ 553
				{ 731
vw	0.900	w	0.988	800
w	0.819	w	0.912	{ 555
				{ 751
		w	0.886	840
		mw	0.806	844

photometer records of photographs of the film as prepared and of a film of $\gamma\text{Al}_2\text{O}_3$ have been made in the Physics Department of the Laboratory. On these records the haloes produced by the material as prepared appear as small humps or steps on a background of rapidly decreasing blackening. The background, which is a scattering effect produced in the film, and forms a part of the diffraction effect, has been removed in an arbitrary manner to facilitate comparison with the ring-pattern. This has been done by plotting the values of $\log I_0/I$ against $\sin \theta/\lambda$, where I_0 is the ordinate from the line of

complete blackening on the microphotometer record to the background line and I is the ordinate from the line of complete blackening to the trace of the record. The average result obtained from opposite sides of two

Fig. 7.



Intensity curves derived from microphotometer records of aluminium oxide films.

photographs is shown in the upper part of fig. 7. The lower part of the same figure is the result of treating a photograph of $\gamma\text{Al}_2\text{O}_3$ in the same way. It will be seen that the outer halo, for which $\sin \theta/\lambda = 0.37$ falls very near

the two fairly strong rings (511), (333) and (440) of $\gamma\text{Al}_2\text{O}_3$. Schmid and Wasserman do not state which rings of $\gamma\text{Al}_2\text{O}_3$ fall in the same region as this outer halo, and give no measurements to make a comparison possible. The inner halo of fig. 7 is evidently not due to the strong (311) of $\gamma\text{Al}_2\text{O}_3$, and could only be assigned to $\gamma\text{Al}_2\text{O}_3$ if some reason could be advanced for supposing that the film was in such a state that the (220) reflexion was of very much greater relative intensity than in the crystalline material.

We have, therefore, the two following reasons for supposing that the film is not composed of very small crystals of $\gamma\text{Al}_2\text{O}_3$. Firstly, the fact that the inner halo does not correspond with any intense ring of the $\gamma\text{Al}_2\text{O}_3$ spectrum; and secondly, the appearance on heating of an intense ring with a spacing, which is not to be found in $\gamma\text{Al}_2\text{O}_3$.

An attempt has been made to apply the method so successfully used by Warren⁽¹¹⁾ in the elucidation of the structure of vitreous silica to the data given by the films. No success has as yet been obtained, but it seems that an analysis on these lines might lead to interesting results.

Summary.

The oxide film found on aluminium at room temperature has been isolated by removing the metal by treatment in HCl gas at 250° C. Electron diffraction photographs of the film show that it is amorphous. Heating the film at temperatures up to 650° C. does not induce crystallization, but at a temperature of 680° C. crystallization begins slowly. The film becomes a random mass of small crystals of cubic $\gamma\text{Al}_2\text{O}_3$, the form of alumina which is present on the surface of molten aluminium.

Acknowledgments.

The work has been carried out for the Executive Committee of the National Physical Laboratory in connexion with an investigation into the structure of oxide films and metal surfaces. We are indebted to Dr. C. H. Desch, F.R.S., Superintendent of the Department of Metallurgy and Chemistry, for his interest in the experiments.

References.

- (1) Annual Report of the National Physical Laboratory, 1933, p. 171.
- (2) Steinheil, A., *Ann. der Physik*, xix, p. 465 (1934).
- (3) Raether, *Z. f. Physik*, lxxxvi. p. 82 (1933).
- (4) Darbyshire and Dixit, *Phil. Mag.* xvi. p. 961 (1933).
- (5) Ehrenfest, *Amsterdam Acad.* xvii. p. 1184 (1915).
- (6) Zernicke and Prins, *Z. f. Physik*, xli. p. 184 (1927).
- (7) Withey and Millar, *J. Soc. Chem. Industry*, xlv. p. 170 (1926).
- (8) Verwey, *Z. f. Kristallographie*, xci. p. 65 (1935).
- (9) Verwey, *Z. f. Kristallographie*, xci. p. 317 (1935).
- (10) Schmid and Wasserman, *Hauszeitschrift der V. A. W. u. d. Erftwerk*, A. G., 4/6, p. 100 (1932).
- (11) Warren, *Z. f. Kristallographie*, lxxxvi. p. 349 (1933).

LII. The Inductance of Iron-cored Coils having an Air Gap. By G. F. PARTRIDGE, B.Sc., F.Inst.P. *

THE design of iron-cored coils having an air gap is of some importance. Choke coils and transformers of this form are being used in eliminators and similar apparatus. The problem of design in these cases is usually complicated by the fact that direct current is passed through the coil superposed on which is the alternating current. The effect of direct current will not be considered in this paper.

There is also a further use for such coils as inductances whose value can be varied by altering the length of the air gap, and the subject of this paper is the determination of the inductance of such a coil to alternating current. It is the well-known problem of the inductance of a concentrated coil on an iron ring. It is generally accepted that when there is no air gap, and provided the flux density is not very large, the inductance is given by

$$L = \frac{0.4\pi \cdot N^2 \cdot \mu A \times 10^{-8}}{l_i} \text{ henries,}$$

where

N = number of turns.

A = area of cross-section of iron in cm.^2 .

l_i = mean length of iron path in cm.

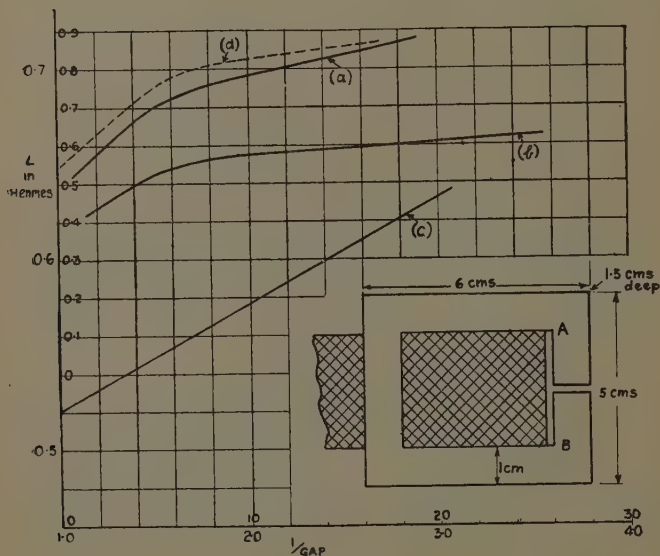
μ = permeability of iron.

* Communicated by the Author.

If the flux density in the iron is high there is leakage of flux in the air space. The exact calculation of the distribution of this leakage flux has never been solved, as it presents many analytical difficulties, even when the permeability of the iron can be regarded as constant, which cannot be assumed under these conditions.

When there is an air gap the leakage of the flux may

Fig. 1.



be considerable even at low-flux densities, and the effect of this on inductance has not been generally recognized.

Measured Value of Inductance.

As illustrating the effect we will take the case of the coil represented in the insert of fig. 1.

The inductance of this coil has been measured by a tuning method as shown in fig. 2.

A is an air-cored transformer, the secondary circuit

of which can be tuned by the condenser C. It can be shown that the condition for tuning is given by

$$L\omega^2 - \frac{1}{C} = \frac{M^2\omega^4}{r^2 + l^2\omega^2}, \quad \dots \quad (1)$$

where

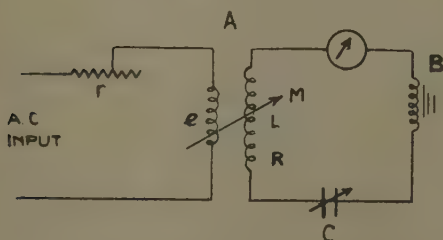
l, r are the inductance and resistance respectively of the primary circuit.

L, R the inductance and resistance respectively of the secondary circuit.

M is the mutual inductance between primary and secondary.

B is the iron-cored coil whose inductance is being measured.

Fig. 2.



It is arranged that the resistance of B is a small portion of the secondary resistance. The inductance of B is determined for any particular current by inserting it in the secondary circuit, meanwhile adjusting the current by means of r . With the same value of r the tuning value of the condenser C is then determined with B out of the secondary circuit.

Under these conditions the value of the right-hand side of equation (1) is constant, and we have

$$L\omega^2 - \frac{1}{C_1} = K \text{ without } B \text{ in circuit,}$$

$$(L + \phi)\omega^2 - \frac{1}{C_2} = K \text{ with } B \text{ in circuit,}$$

from which we have

$$\text{Inductance of } B = \phi = \frac{C_1 - C_2}{C_1 \cdot C_2 \cdot \omega^2}, \quad \text{where } \omega = 2\pi f,$$

$$= 2\pi \times \text{frequency of alternating current.}$$

The inductance for various lengths of gap was determined in this way, and the values are given in the following table.

TABLE I.
Variation of Inductance with Gap.
Frequency 172-174.5 pps.

L in henries } 0.555 0.586 0.609 0.639 0.672 0.774 0.853 0.917
Gap = l_a cm. } 0.635 0.457 0.394 0.286 0.239 0.108 0.061 0.045
1/gap..... 1.58 2.19 2.54 3.50 4.18 9.25 16.4 22.2

In fig. 1 the inductance has been plotted against the reciprocal of the length of gap—see curve (a). It will be seen that for fairly large gaps the curve has a constant slope. (This portion of the curve has been plotted on a larger scale—see curve (c).)

The initial portion of the curve can therefore be represented by the expression

$$\left. \begin{aligned} L &= A + \frac{K}{\text{gap}} \\ &= 0.472 + \frac{0.054}{\text{gap}} \quad \text{as determined from the curve.} \end{aligned} \right\} \quad (2)$$

The general expression for the calculation of inductance is

$$L = \frac{0.4\pi \cdot N^2 \cdot A \times 10^{-8}}{\frac{l_i}{\mu} + l_a} \text{ henries}$$

$$= \frac{1.25 N^2 A \times 10^{-8}}{\frac{l_i}{\mu} + l_a} \text{ henries, ... } (3)$$

where

l_i = the mean length of the iron path in cm.

l_a = the length of the air gap in cm.

Now from equation (3) we have, inserting the values of $N=1700$, and $A = 1.5 \text{ cm.}^2$;

$$L = \frac{0.0542}{\frac{l_i}{\mu} + l_a}$$

$$= \frac{0.0542}{l_a},$$

since $\frac{l_i}{\mu}$, as it will appear later, is negligible compared with l_a when the latter lies between 0.3–0.6 cm.

We thus see that the variation of inductance with the length of gap at the initial part of the curve agrees with that given by the theoretical expression of equation (3), but that the measured value of the inductance is greater than that given by theory by 0.472 henry. Thus for a gap of 0.394 cm. the calculated inductance is 0.137 henry, whereas the measured value is 0.609 henry.

Leakage Flux.

The additional inductance 0.472 henry, although much larger than anticipated, must be due to leakage flux. In order to investigate this, some of the turns were removed from the winding, leaving a space around the limbs of the gap around which a small search coil could be wound. The dimensions of the winding, etc., were then as shown in fig. 3, the number of turns being now 1410.

The relationship between inductance and gap as determined by measurement is shown in curve (b), fig. 1, and the equation for the initial part of the curve is

$$L = 0.353 + \frac{0.37}{l_a} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The theoretical value from equation (3) is

$$L = \frac{1.25 \times 1410^2 \times 1.5 \times 10^{-8}}{\frac{l_i}{\mu} + l_a}$$

$$= \frac{0.372}{l_a}.$$

Using the search coil, the flux along the limb and in the gap was determined at the various positions shown in fig. 3. From the values obtained the leakage flux at each position was determined by differences. These are given in Table II.

The total leakage over the portion measured, being the sum of the above, is 1700. The flux across the gap was 1015.

It will now be shown that the leakage flux can be estimated with a fair degree of accuracy.

Fig. 3.

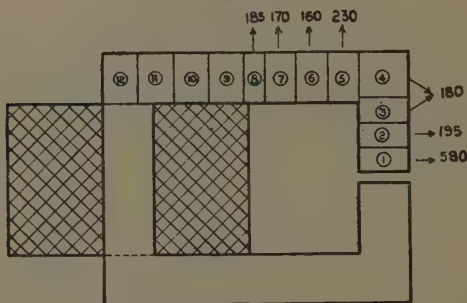


TABLE II.

Leakage Flux measured,
current in winding 0.09 amp.

Position	(1)	(2)	(3 & 4)	(5)	(6)	(7)	(8)
Leakage flux in lines per cm. ² }	580	195	180	230	160	170	185

The total flux linking the coil winding can be expressed as

$$F = \frac{1.25NI}{\frac{l_i}{\mu A_i} + \frac{l}{A}}$$

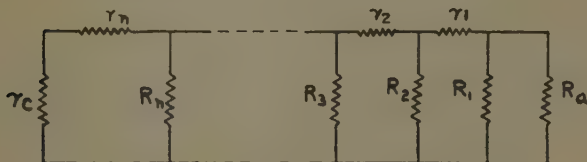
$$= \frac{158}{r + R}$$

since

$$N = 1410; \quad I = 0.09 \text{ amp.},$$

where l_i , A_i are the length and area of the iron path and l , A the length and area of the equivalent air paths. R and r are termed the reluctance*, and are of the nature of resistance to magnetic flux. If now the iron path is divided off into sections, as in fig. 3, then the total reluctance of the path of the flux can be represented by the resistance diagram of fig. 4, where r_c is the reluctance of the iron core, r_1 , r_2 , etc. the reluctances of the iron in the various sections, R_1 , R_2 etc. the reluctances of the air-leakage paths, and R_a the reluctance of the air gap. The reluctances r_1 , r_2 etc. will be in general very small compared with their corresponding air-leakage paths R_1 , R_2 etc. We may regard them as

Fig. 4.



negligible for the present when we see that the total reluctance of the circuit is given by

Total reluctance } $= r_c +$ equivalent reluctance of all the air paths R_1 , R_2 , etc., in parallel with the gap R_a

$$\begin{aligned}
 &= r_c + \frac{1}{\text{sum of a series of conductances } \frac{1}{R_1} \text{ etc.}} \\
 &= r_c + \frac{1}{\sum \frac{l}{A}},
 \end{aligned}$$

since $\frac{1}{R} = \frac{A}{l}$.

* According to the notation usually adopted by engineers, the reluctance is $\frac{10}{4\pi} \cdot \frac{l}{A}$. In order to avoid the repetition of the constant $\frac{10}{4\pi}$ the reluctance has been taken as $\frac{l}{A}$ (see 'Electromagnetic Problems in Engineering,' by Hague, pp. 39-41).

We have thus to find the sum of all the values of $\frac{A}{l}$ appropriate to each leakage path, together with that of the air gap.

To evaluate this the exposed areas of each section of the iron are given in Table III., and also the shortest air path between each section and its corresponding section on the other limb of the iron. From these the appropriate values of $\frac{A}{l}$ have been calculated.

The leakage flux from each section has then been calculated ($F=158 \times \frac{A}{l}$, assuming r_s also small).

TABLE III.
Leakage Flux calculated.

Section.	Air gap.	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.
Area=A	1.5	2.3	2.3	2.3	5.0	3.1	3.1	3.1	2.0	2.5	2.5	2.5	4.0
Length of path= l } ..	0.239	0.7	1.62	2.54	3.0	3.0	3.0	3.0	3.0	3.0	5.5	6.5	6.7
A/l	6.27	3.30	1.40	0.91	1.67	1.03	1.03	1.03	0.67	0.83	0.46	0.38	0.60
F	1000	520	220	144	264	162	162	162	105	131	72	60	95

Comparing the calculated values with the measured as given in Table II., we see that the agreement is good. The agreement is better shown by comparing the measured values with the calculated for the flux across the gap and the total leakage between positions (1) and (8). These are as follows :—

	Leakage flux (1) to (8).	Flux across gap.
Calculated	1739	1000
Measured	1700	1015

It will be seen from the results of Table III. that the greater part of the leakage of flux occurs near the gap.

The value of $\xi \frac{A}{l}$ for this part can be determined more exactly by integration.

It can be shown that for the limbs of the gap AB in fig. 1

$$\begin{aligned}\xi \frac{A}{l} &= \int_{l_a}^{S/2} \frac{P}{2} \frac{dx}{x} \\ &= \frac{P}{2} \log_e \frac{S}{l_a},\end{aligned}$$

where

P=perimeter of the iron=5 cm.

S=length AB =3 cm.

l_a =length of gap =0.239 cm. in this case.

Hence $\xi \frac{A}{l} = 6.32$ as compared with 5.61 obtained by taking sections. Taking this more correct value, we find that for the leakage paths.

$$\xi \frac{A}{l} = 14.02.$$

The total reluctance of the circuit is therefore given by

$$\begin{aligned}\text{Total reluctance} &= r_c + \frac{1}{\left[14.02 + \frac{A}{l} \text{ gap} \right]} \\ &= \frac{1}{\left[14.02 + \frac{A}{l} \text{ gap} \right]},\end{aligned}$$

if r is negligibly small.

Calculation of Inductance.

We can now see the significance of the terms of the equation (4) derived for the initial straight portion of the curves in fig. 1.

As shown above,

$$L = \frac{1.25 \cdot N^2 \times 10^{-8}}{\text{reluctance of circuit}}$$

$$\begin{aligned}
 &= 1.25 \cdot N^2 \times 10^{-8} \left(14.02 + \frac{A}{l} \text{ gap} \right) \\
 &= 0.351 + \frac{0.372}{l_a},
 \end{aligned}$$

since $N=1410$ and $A=1.5$ cm.²

The constant term is therefore the inductance produced by the leakage flux linking the coil. The leakage paths depend upon the shape of the iron and the size of the coil, and are approximately independent of the length of the air gap for large gaps provided the reluctance of the iron can be neglected, so that, as we see from the above, the "leakage inductance" is a constant quantity.

In order to compare the curves (*a*) and (*b*) obtained respectively for the large coil (1700 turns) filling the window of the stampings, and for the small coil (1410 turns) used in the flux measurements, the ordinates of the curve (*b*) have been multiplied by the square of the ratio of the turns and the results plotted—see dotted curve (*d*), fig. 1.

It will be seen that the curve (*d*) lies above (*a*), the slope at the beginning being the same as (*a*), as would be anticipated.

The fact that the curve (*d*) lies above (*a*) indicates that the gap with the smaller coil is of less effective length, due to greater leakage flux, which also agrees with what would be expected for a smaller coil.

Thus from the equation (2) for the larger coil the leakage conductance $\frac{A}{l}$ is found to be 13.0 as compared with 14.02 for the smaller coil.

We see, therefore, that for gaps not less than 0.2 cm. length the inductance is given very nearly by

$$L = 1.25 N^2 \times 10^{-8} \left(\frac{A}{l} \text{ leakage} + \frac{A}{l} \text{ gap} \right). \quad (5)$$

With smaller gaps the reluctance of the iron is no longer negligible. To calculate this according to the scheme suggested would be very laborious. A good approximation, however, can be obtained by assuming that the leakage paths shunt the gap, and that the reluc-

tance of the whole of the iron is in series with this. We then have

$$L = \frac{1.25N^2 \times 10^{-8}}{\frac{l_i}{\mu A_i} + \frac{1}{\xi \frac{A}{l}}} = \frac{1.25N^2 \times 10^{-8}}{\frac{l_i}{\mu A_i} + \frac{1}{\left[14.02 + \frac{A}{l} \text{ gap}\right]}}$$

$$F = \frac{1.25NI}{\frac{l_i}{\mu A_i} + \frac{1}{\left[14.02 + \frac{A}{l} \text{ gap}\right]}}$$

Taking a gap of .05 cm. and l_i as 18 cm. we have for the smaller coil

$$L = \frac{0.0248}{\frac{12}{\mu} + .0227},$$

$$F = \frac{158}{\frac{12}{\mu} + .0227}.$$

From the second expression, knowing the B/H curve of the iron, μ has been found to be 650, and inserting this in the expression for L, we find that the inductance is 0.60 henry. From the curve (b), fig. 1, the measured value of the inductance for this gap is 0.61 henry.

Gap in Core of Coil.

In the foregoing example we have calculated the inductance when the air gap is outside the core of the coil. With the gap inside the core of the coil it might be expected that leakage would be negligible, and that the inductance would be given by equation (3), taking l_a as the actual length of the gap and using the appropriate value of permeability.

It has been found experimentally that this is not the case, but that leakage around the gap must still be allowed for. This is due to the fact that the winding cannot be made to fit exactly round the iron core, so that there is always an air space. In addition there is the space occupied by the insulation of the coil. The

value of $\xi \frac{A}{l}$ for the leakage path across the gap will in this case be given by

$$\xi \frac{A}{l} = \frac{P}{2} \log_e S.$$

To take an example. The stampings of the coil previously considered were reversed so as to give a gap of 0.635 cm. in the centre of the core, the dimensions being as before.

From these figures we find that

$$\frac{A}{l} \text{ gap} = 2.36 \quad \frac{A}{l} \text{ leakage} = 3.88.$$

The net reluctance of the gap is, therefore, the reciprocal of the sum of these values, i. e., $\frac{1}{6.24} = 0.16$.

We therefore have

$$L = \frac{0.0248}{\frac{12}{\mu} + 0.16}$$

and

$$F = \frac{158}{\frac{12}{\mu} + 0.16}.$$

The value of μ given by the second expression is 270, and inserting this in the first expression we find the value of L to be 0.122 henry. The measured value was 0.127 henry.

It thus appears that a knowledge of the leakage reluctance shunting the gap is of vital importance in design. In the present case, if leakage is neglected the calculated value for the inductance is very much in error, being 0.51 henry.

General Considerations.

It will be obvious that a general formula for the inductance of iron-cored coils having an air gap cannot be given to cover all cases, since the leakage reluctance depends very much upon the shape of the iron in the circuit.

If the reluctance of the iron can be neglected in comparison with that of the air paths it has been found that the variation of inductance with length of gap is given by an expression of the form (see equation (5))

$$L = \frac{1.25N^2A \times 10^{-8}}{B + l_a} \text{ henries, ... (6)}$$

where B represents the effect of leakage.

A formula has been given by Nottage * for small air gaps of the form

$$L = \frac{1.25N^2.A.K \times 10^{-8}}{l_a} \text{ henries, . . . (7)}$$

where K is a constant for any particular shape of iron circuit.

Comparing (6) and (7) it will be seen that the second expression assumes that the leakage reluctance is $\frac{K}{A}$ times the reluctance of the gap. The preceding results show, however, that the leakage reluctance is not proportional to gap length even when the gap is relatively small.

From the figures obtained in the section on leakage flux it will be seen that as the length of the limb in which the gap occurs is increased the reluctance of the leakage paths other than around the limb containing the air gap is proportionately reduced. If this limb is equal to, or greater in length than, the other limb of the window in the stampings, it has been found that the inductance is given to within 10 per cent. by taking as the leakage reluctance only that portion occurring across the gap, for all gaps up to a ratio of 0.02 where gap ratio is defined as $\frac{l_a}{l_i}$:

The total reluctance R of the air path is then given by

$$\frac{1}{R} = \frac{A}{l_a} + \frac{P}{2} \log_e \frac{S}{l_a}$$

* See 'Calculation of Inductance and Capacity,' by W. H. Nottage, p. 28.

and the inductance by

$$L = \frac{0.4\pi \cdot N^2 \times 10^{-8}}{\frac{l_i}{\mu A_i} + R} \text{ henries.}$$

The same expression will also give the inductance when the air gap is inside the core of the coil to within 2 per cent.

Summary.

Curves have been given showing the variation of inductance of an iron-cored coil with length of air gap. For relatively large gaps the inductance varies according to the well-known expression

$$L = \frac{1.25 N^2 A \times 10^{-8}}{l_a} \text{ henries,}$$

but, due to leakage, the measured value of the inductance may be much larger than that given by the above expression.

A method of estimating the effect of leakage has been given sufficient to enable the inductance to be calculated within a few per cent.

A general formula has been given for calculating the inductance when the window of the iron stampings is rectangular and the air gap is cut in the longer side, sufficient to give the inductance within 10 per cent. for all gaps up to a gap ratio of 0.02.

The same formula gives the inductance within two per cent. when the air gap is introduced into the core of the iron circuit.

The author wishes to express his indebtedness to the Admiralty for permission to publish this paper.

LIII. Slow Viscous Flow and Elastic Deformation.

To the Editors of the Philosophical Magazine.

GENTLEMEN,—

THE recent papers of Gemant* and Yates-Fish† on dipole theory, and the 'First Report on Viscosity and Plasticity' of the Committee of the Amsterdam

* Phil. Mag. (7) xix. p. 746 (1935).

† Phil. Mag. (7) xxi. p. 226 (1936).

Academy of Sciences (1935), suggest that full advantage is not being taken of the analogy between slow steady viscous motion and elastic deformation. This analogy enables us to take over from the theory of viscous flow solutions of the elastic problems arising in the dipole theory, and to take over from elasticity formulæ and observations of which the theory of viscosity measurement stands in need. Full appreciation of it would also prevent further duplication of analysis in the solution of analogous problems in the two theories.

The analogy is apparent on comparison of the general equations of the two theories. In those of viscous motion the inertia terms are neglected and "viscous incompressibility" * is assumed. In those of elasticity elastic compressibility must usually, but (as is illustrated below) not always, be assumed. The analogy might be formally stated thus:—"Let there be two isotropic and homogeneous bodies of the same shape, one purely viscous, the other purely elastic and incompressible. The two are either loaded in the same way, or, if certain displacements are prescribed as given functions in the elastic body, then the corresponding velocities in the viscous body are prescribed as the same functions. Let the *displacements* of the incompressible elastic body be expressed as certain functions of x, y, z , and the modulus of rigidity μ (for the incompressible solid Poisson's ratio σ is $\frac{1}{2}$ and Young's modulus is 3μ). Then the *velocities* of the viscous body are expressed by the same functions when the coefficient of viscosity η is substituted for μ as long as the deformation makes no significant change in the boundaries." The converse will also be true.

In certain types of viscous flow, notably in the motions due to rotation of solids of revolution about their axes, in infinite fluid, the pressure in the fluid remains constant †. It is easily verified that in this case the solutions are also solutions for the *compressible* elastic solid, since they make the elastic dilatation zero and satisfy the equations independently of the compressibility. Thus Stokes' formula for the steady rotation of a sphere in a fluid is valid for the small rotational displacement of a rigid sphere imbedded in an elastic medium, and a similar

* *I. e.*, a hydrostatic pressure does not give rise to any *velocities*.

† G. B. Jeffery, Proc. Lond. Math. Soc. (2) xiv. p. 327 (1915).

adaptation may be made of Jeffery's formulæ* for ellipsoids of revolution and two spheres.

Two of the methods in use for measuring high viscosity involve the stretching of a fibre and the bending of a beam. The analogy indicates at once that the formulæ for these tests may be derived by methods parallel to those used in elasticity to establish stretching and bending formulæ. In the stretching test, for instance, corresponding to the elastic displacement Pl/AE (in the customary notation), the velocity of the end of the viscous fibre will be $Pl/3A\eta$. This is a very simple case, but the bending tests are not as simple, and suitable formulæ do not seem to have been given. The 'Report' describes (pp. 149-150) a beam test in which the thickness of the specimen is of the same order as the effective length. From what is known of the elastic behaviour of a beam of such proportions we should expect that the sag would be roughly equally due to shearing strain and to flexure—i. e., pure extension of filaments below the neutral plane and pure contraction above. Since the shearing deformation cannot be evaluated with any accuracy by simple formulæ it is not to be expected that such a beam could provide a satisfactory measure of the viscosity. It would appear necessary to use a slender beam and rely entirely on flexural deformation. The well-known formulæ of the strength of materials for the various modes of loading and supporting are then available as soon as Young's modulus is replaced by 3η . The elasticity theory as it is at present developed includes numerous other solutions which have interesting analogues as, for instance, the deformation of bodies under their own weight and the Boussinesq problem of the rigid die bearing on a flat surface†. The latter gives a velocity $P/8a\eta$ for a die of radius a under a load P when the tangential forces are neglected.

The singularity known as the "centre of dilatation" in elasticity can be adapted in formulating the law of expansion of a bubble in a very viscous fluid. The radial displacement due to pressure P_1 in a spherical cavity of radius a is $P_1 a^3/4\mu r^2$. In the fluid the effective boundary

* G. B. Jeffery, *loc. cit.*

† See, for instance, H. Lamb, *Proc. Lond. Math. Soc.* xxxiv. p. 276 (1902).

pressure causing flow is $P - P_0 - 2T/a$, where P is the gaseous pressure, P_0 the external pressure on the fluid, and T the surface tension. If the bubble contains a definite amount of gas Boyle's law requires $P = ka^{-3}$, and we are thus led to the following equation for the velocity of expansion,

$$\frac{da}{dt} = \frac{1}{4\eta} \left(\frac{k}{a^2} - P_0 a - 2T \right),$$

which is easily integrated if required. If the external pressure on a viscous mass containing a bubble were changed the size of the bubble would change slowly in accordance with this equation, and the whole mass would show a corresponding change in volume, approaching an equilibrium volume. The mass would behave, therefore, like a compressible plastic material with finite strength, although the material itself is purely viscous and incompressible. Under non-hydrostatic forces it would, of course, have no finite strength.

Yours faithfully,
J. N. GOODIER.

Ontario Research Foundation,
May 2nd, 1936.

LIV. *The Specific Resistance and Temperature Variation of Resistance of Nickel and some of its Alloys in the Annealed and Unannealed States.* By I. H. THOMAS, B.Sc., and R. M. DAVIES, M.Sc., F.Inst.P., University College of Wales, Aberystwyth*.

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* Communicated by Vice-Principal Gwilym Owen, M.A., D.Sc.
Phil. Mag. S. 7. Vol. 22. No. 148. Oct. 1936. 2 Y

I. INTRODUCTION.

A PART from the intrinsic and technical interest of the numerical results obtained an investigation of the specific resistance and temperature variation of resistance of nickel and some of its alloys in the annealed and unannealed states should yield results which might be of interest in connexion with the following problems:—

(a) The relation between the specific resistance and the temperature coefficient of resistance of alloys in the annealed and unannealed states.

(b) The change in resistance which occurs when a ferromagnetic substance changes into the paramagnetic form at the Curie temperature. (Nickel and some of the alloys investigated are ferromagnetic at ordinary temperatures.)

It will be convenient at this stage to consider modern views on the theoretical aspects of these two problems.

(a) *Relation between Annealed and Unannealed Alloys.*

For simplicity we may confine ourselves to the consideration of solid binary alloys. X-ray analysis has shown that in a cast (or unannealed) alloy the two kinds of atoms are arranged at random on a common space-lattice. When such alloys are subjected to prolonged annealing it is found that, for alloys of certain compositions, marked changes occur in some of their physical properties. Thus, changes in electrical resistance, Hall coefficients, thermoelectric potential differences against other metals or alloys, and, for magnetic alloys, changes in magnetic susceptibility have been observed.

In order to explain in particular the change in electrical resistance due to annealing Tammann ⁽¹⁾, in 1919, put forward the hypothesis that on annealing the atoms segregate into definite positions on the space-lattice, as opposed to a random distribution in an unannealed alloy.

The reality of this segregation into definite positions has been demonstrated by Johansson and Linde ⁽²⁾, using X-ray methods, for the copper-gold, copper-palladium, and copper-platinum systems.

For example, alloys containing 75 per cent. (atomic) of copper consist in the unannealed state of the two kinds of atoms arranged at random on a face-centred

cubic lattice. After prolonged annealing the atoms rearrange themselves, so that the atoms at the centre of the cube faces are copper while those at the cube corners are of the other metal, thus forming what is known as a superlattice.

A more complex system has been studied by Bradley and Jay ⁽³⁾, their conclusions being substantially the same as those of Johansson and Linde.

The relationship existing between alloys of the same composition in the two states mentioned may be derived from energy considerations. An ordered arrangement of atoms in a lattice must have a slightly lower potential energy than the random distribution, and since the state of lowest potential energy is the most stable state, there will be a definite tendency for an unannealed alloy to proceed to the annealed state. Though this process may be so slow as to be inappreciable at room-temperatures, it may be accelerated by suitable heat treatment.

On this theory the effect of annealing on the electrical resistance of alloys may be briefly described as follows. Modern theories of metallic conduction state that the electrical resistance of a pure metal or an alloy possessing a pure superlattice is caused solely by the scattering of electron waves by atoms (or ions) in their space lattices. In the hypothetical case of the atoms being at rest there would be no scattering, resulting in infinite conductivity or zero resistance. At low temperatures the amplitudes of the vibrations of the atoms are small, and also the mean position of the atoms is independent of temperature, since the atomic movements are purely sinusoidal; this results in a small change of resistance with temperature. With increasing temperature the amplitude of oscillation increases, and the oscillations become asymmetric, more scattering ensues, and the electrical resistance increases.

Thus for a pure metal the relation between resistance and temperature may be expressed approximately by

$$R_0 = K_0 T,$$

where

R_0 = the resistance at any temperature $T^\circ \text{ K.}$,

and

K_0 = a constant involving the electronic charge, etc.

Electrical resistance, however, can be created not only by the thermal agitation of the atoms but also by disorder, so that in an unannealed alloy the resistance consists of two parts, one due to the thermal agitation and one due to the disorder, the resistance due to this second effect being determined by the degree of disorder present.

Thus for an unannealed alloy the resistance-temperature relation may, to a first approximation, be expressed

$$R_1 = R_A + K_1 T,$$

where R_1 is the alloy resistance at a temperature $T^\circ \text{K}$, R_A the resistance created by the disorder, and K_1 a constant.

Let ρ_0 and ρ_1 represent the specific resistances at the same temperature of a given alloy in the annealed and unannealed states respectively. Let ρ_A denote the specific resistance due to disorder.

Then

$$\rho_0 = K_0' T \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$\rho_1 = \rho_A + K_1' T. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Subtracting (1) from (2),

$$\rho_1 - \rho_0 = \rho_A + (K_1' - K_0') T.$$

Hence $(\rho_1 - \rho_0)$ may be positive or negative, depending on the signs and numerical magnitudes of ρ_A , K_1' , K_0' , and T . In general K_0' and K_1' are positive, so that $(\rho_1 - \rho_0)$ is determined by ρ_A and the magnitudes of K_0' and K_1' .

In the case of the alloys investigated in the present work it is found that at about 20°C . approximately half the alloys have a greater specific resistance in the unannealed state than in the annealed state, whilst the converse holds for the remainder.

(b) *Changes at the Curie Point.*

It has been known for some time that ferromagnetic substances exhibit peculiar changes of resistance on passing through their Curie points. Thus Gerlach and Schneiderhan⁽⁴⁾ investigated the variation in resistance of a nickel wire in the absence of a magnetic field. For temperatures higher than the Curie point the resistance varied linearly with temperature like any other non-

magnetic metal. Below the Curie point, however, the resistance at any given temperature was that of the non-magnetic metal decreased by an amount proportional to the square of the spontaneous magnetization (σ^2). Mathematically, Gerlach's results may be expressed by the equation

$$R_T = R_{NT} - C\sigma^2, \quad (3)$$

where C is a constant independent of temperature.

R_T = the resistance of the nickel wire at any temperature.

R_{NT} = the resistance of the normal metal at the same temperature, and obtained for temperatures below the Curie point by extrapolation from the results obtained above the Curie point, making the assumption that the resistance of a normal, non-magnetic metal varies linearly with temperature.

Then

$$\frac{dR_T}{dT} = \frac{dR_{NT}}{dT} - C \frac{d\sigma^2}{dT} (4)$$

For $T > T_C$ (T_C = Curie temperature) equation (4) becomes

$$\frac{dR_T}{dT} = \frac{dR_{NT}}{dT}, \text{ since } \frac{d\sigma^2}{dT} = 0,$$

and for $T < T_C$ equation (4) applies.

Bates⁽⁵⁾ investigated the ferromagnetic compound manganese arsenide (MnAs), which has a very low Curie point (c. 42° C.). These peculiar changes of resistance were observed by him with all the specimens examined.

The actual process that gives rise to these characteristic changes is still somewhat obscure. While changes in magnetic energy on passing from a ferromagnetic to a paramagnetic state have some influence on the change, it is believed that this does not wholly account for the anomalies present in the neighbourhood of the Curie point.

Two types of changes will in general cause an electrical resistance-temperature curve to show anomalies and singular points.

(i.) Where properties (more specifically in our case electrical resistance) below and above a transition tem-

perature alter normally with temperature but discontinuously at the transition temperature. These are what may be termed Gibbs phase changes. They involve recrystallization at the transition temperature, accompanied by change in volume and in entropy, *i. e.*, by the absorption or evolution of heat at the transition temperature.

(ii.) Where the changes occur without recrystallization and the physical properties alter abnormally but continuously in a temperature interval. Weaver ⁽⁶⁾ was the first to point out that the change at the Curie point for ferromagnetics involved no recrystallization. Ehrenfest proposes to call this kind of transformation "a phase change of the second order"; he shows that transitions from the ferromagnetic to the paramagnetic state at the Curie point are examples of this transformation. Further examples are the transitions of certain ammonium salts, solid halogen hydrides, solid methane, and the transition from ordinary conductivity to superconductivity. The criterion for this class appears to be that the abnormal changes in the physical properties start at a temperature $T < T_1$, and are completed at a temperature $T > T_2$, where (T_1, T_2) is a small temperature interval bounding, in the case of the ferromagnetic-paramagnetic transition, the Curie temperature.

Bates (*loc. cit.*), on the basis of his work on manganese arsenide, arrives at the conclusion that two processes are responsible for the peculiar resistance changes at the Curie temperature:

(1) Changes in resistance due to changes in magnetic energy on passing from a non-magnetic to a magnetic state.

(2) Changes due to the formation of fissures and their enlargement during transitions from the paramagnetic to the ferromagnetic state.

It appears that once the manganese arsenide has had its temperature raised above the Curie temperature certain permanent changes are produced, and the resistance-temperature curve obtained at the first heating is not reproducible when the specimen is cooled to 0°C . and heated again.

II. THE ALLOYS INVESTIGATED.

A description of the alloys to be investigated, together with short notes on the annealing of the "soft" specimens, is appended below. All "soft" specimens were canister annealed, *i. e.*, while in the furnace they were packed in air-tight containers. After annealing they were allowed to cool slowly to room-temperature. In each case annealed and unannealed specimens were investigated.

(i.) *Nickel.*

Purity : 99.2 per cent. nickel.

Nature : ferromagnetic, with a Curie point at about 350° C.

Annealed at 850° for 45 hours.

(ii.) "*Monel.*"

Composition : 67 per cent. Ni ; 28 per cent. Cu ; balance Fe and Mn.

Nature : ferromagnetic, with a Curie point at about 100° C.

Annealed at 850° C. for 45 hours.

(iii.) 48 per cent. *Nickel-Iron.*

Composition : 48 per cent. Ni ; 52 per cent. Fe.

Nature : ferromagnetic, with a Curie point at about 425° C.

Annealed at 900° C. for 7 hours. This alloy has a very high permeability in weak magnetic fields.

(iv.) "*Ferry.*"

Composition : 44 per cent. Ni ; 56 per cent. Cu.

Nature : non-magnetic.

Annealed at 850° C. for 45 hours.

(v.) "*Brightray.*"

Composition : 80 per cent. Ni ; 20 per cent. Cr.

Nature : non-magnetic.

Annealed at 900° C. for 7 hours.

(vi.) "*Glowray.*"

Composition : 65 per cent. Ni ; 15 per cent. Cr ; 20 per cent. Fe.

Nature : ferromagnetic.

Annealed at 900° C. for 7 hours.

(vii.) "*Dullray.*"

Composition : 34 per cent. Ni ; 4 per cent. Cr ;
62 per cent. Fe.

Nature : non-magnetic.

Annealed at 900° C. for 7 hours.

(viii.) 20 per cent. *Nickel-Silver.*

Composition : 62 per cent. Cu ; 18 per cent. Zn ;
20 per cent. Ni.

Nature : non-magnetic.

Annealed at 650° C. for $\frac{3}{4}$ hour.

The above specimens were presented to the laboratory for the purpose of this investigation by Messrs. Henry Wiggin and Company, Ltd. The annealing had been done by the donors before they sent the specimens.

III.

(1) *The Measurement of Specific Resistance
at Room-temperature.*

For the specific resistance determinations the alloys used were in the form of thin rectangular strips, approximately 10 cm. long, 0.6 cm. wide, and 0.05 cm. thick.

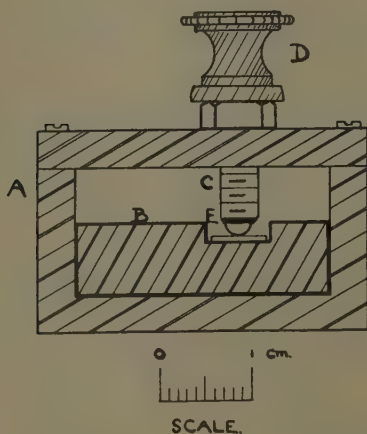
The resistance measurements were carried out by a potentiometer method, and the specimens had therefore to be fitted with current and with potential leads. The current leads were soldered to the ends of the specimen, whilst the potential leads were connected to two terminals of the type shown in fig. 1.

In this diagram A represents a rectangular brass frame fitted with a brass screw C, to the lower end of which is soldered a hard steel ball-bearing. B represents an ebonite strip 8 cm. long, 4 cm. wide, and 1 cm. thick, in which a groove E, 0.6 cm. wide and 0.1 cm. deep, is cut. The alloy specimen fits tightly into this groove, and the position of the groove is arranged so that its mid-line is vertically under the tip of C when B is placed in position in the frame A. The potential leads are connected to the frame by the screw D.

In carrying out the experiment the specimen under test is placed in the groove E. The two potential terminals are then slipped over B, at a convenient distance apart, and the alloy is kept firmly in position by tightening the screws C.

The resistance of this length of alloy is determined as described below. The "length" of the specimen can be determined accurately, since small circular indentations are made in the alloy by the balls when the screws are tightened, and the distance between the centres of these indentations can be accurately determined by a travelling microscope. For any one specimen three different lengths are taken by moving the potential terminals, the resistance of each determined, and the resistance per unit length evaluated. From a knowledge

Fig. 1.



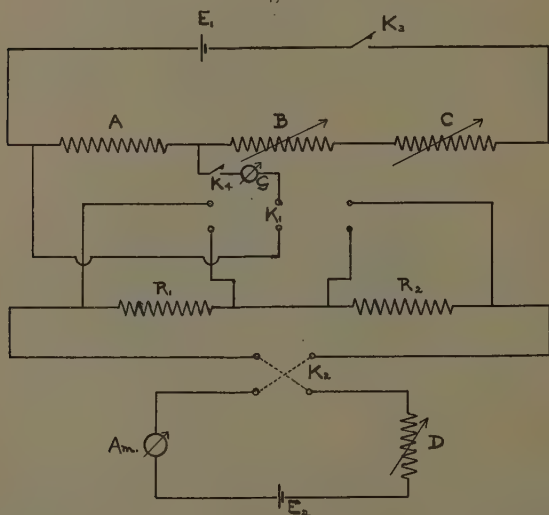
of the width and thickness of the specimen the specific resistance can be calculated.

Preliminary experiments showed that in order to obtain consistent values of the resistance per unit length the potential terminals should not be too near the points where the current enters the specimen. This is due to the fact that near the current terminals the lines of current-flow in the specimen are not parallel to the length of the specimen.

The potentiometer method used for determining the resistance is shown in fig. 2, in which R_1 represents the resistance under test and R_2 a standard 0.001 ohm resistance. These two resistances are connected in series across a 2 volt accumulator E_2 , a rheostat D , and a

reversing key K_2 . A represents a standard 1 ohm resistance, whilst B and C are calibrated resistance boxes; A, B, and C are connected in series across a 2 volt accumulator E_1 , using leads of very low resistance. The potential drops across R_1 and R_2 are successively applied across the standard resistance A by means of the double-pole double-throw switch K_1 , and balance is obtained in each case by variations of B and C. Since the resistance of the leads in the A, B, C circuit is negligible in comparison

Fig. 2.



with the sum of the resistances of A, B, and C, the ratio of the resistance of R_1 to R_2 is inversely proportional to the sum of the resistances A, B and C at balance in each instance.

In order to eliminate small variations in the current in the circuits the procedure adopted consists in balancing first the potential difference across the standard resistance, then that across the alloy specimen, and finally that across the standard resistance; the mean of the two potential differences across the standard resistance is taken as the true value.

Thermoelectric electromotive forces are eliminated by reversing the currents in both circuits and repeating the above procedure. The mean of the two ratios is taken as the true ratio of the resistances of alloy and standard.

(2) *Accuracy of the Results.*

The quantities determined experimentally are :—

- (a) The resistance.
- (b) The dimensions of the alloy specimens.

(a) All the resistance boxes used were calibrated in terms of an N.P.L. calibrated box, and are accurate to better than 0.1 per cent. The resistance of the specimen could be determined in terms of the standard 0.001 ohm resistance—known to 1 in 10,000-(N.P.L.) to better than 0.1 per cent. Hence the actual resistance is accurate to within 0.1 per cent.

(b) The length of the specimen is of the order of 2 or 3 cm., and could be determined to 0.1 per cent. ; the breadth is of the order of 0.6 cm., and could be determined to 1 in 500, whilst the thickness is of the order of 0.05 cm., and could be determined to 1 in 200. Thus the dimensions are accurate to about 1 in 200, or $\frac{1}{2}$ per cent.

Combining (a) and (b), the actual specific resistances are thus accurate to about 0.5 per cent.

(3) *Numerical Results for the Specific Resistance at 19.0° C.*

The method used to calculate the specific resistance may be illustrated by giving the calculation in detail for one of the specimens : “Dullray Soft.”

Let the total resistance in the potentiometer circuit be R_s when the potential difference across the standard low resistance is balanced against the potential difference across the standard ohm resistance (v. fig. 2).

Let the total resistance in the potentiometer circuit be R_A when the potential difference across the alloy is balanced against the potential difference across the standard ohm resistance.

Also, let the distance between the potential contacts be L . If S be the value of the standard low resistance

($S=0.001 \omega$), A the area of cross-section of the specimen, and ρ its specific resistance, then

$$\rho = \frac{R_s}{R_A} \cdot \frac{1}{L} \cdot SA.$$

For "Dullray Soft" the following numerical results were obtained :—

$$(1) L=3.125 \text{ cm.} \quad R_s=1209 \text{ ohm.} \quad R_A=215.5 \text{ ohm.}$$

$$\frac{R_s}{R_A} = 5.610.$$

When the currents in the circuits were reversed :

$$R_s=1212 \text{ ohm.} \quad R_A=215.8 \text{ ohm.}$$

$$\frac{R_s}{R_A} = 5.616.$$

Mean value of

$$\frac{R_s}{R_A} = 5.613.$$

Whence

$$\frac{R_s}{R_A} \cdot \frac{1}{L} = 1.798.$$

$$(2) L=2.479 \text{ cm.} \quad R_s=1211 \text{ ohm.} \quad R_A=271.0 \text{ ohm.}$$

$$\frac{R_s}{R_A} = 4.468.$$

For the currents reversed :

$$R_s=1202 \text{ ohm.} \quad R_A=269.3 \text{ ohm.}$$

$$\frac{R_s}{R_A} = 4.463.$$

Mean value of

$$\frac{R_s}{R_A} = 4.466.$$

Whence

$$\frac{R_s}{R_A L} = 1.800.$$

$$(3) L=1.989 \text{ cm.} \quad R_s=1202 \text{ ohm.} \quad R_A=335.9 \text{ ohm.}$$

$$\frac{R_s}{R_A} = 3.581.$$

For the currents reversed :

$$R_s=1216 \text{ ohm.} \quad R_A=339.1 \text{ ohm.}$$

$$\frac{R_s}{R_A} = 3.584.$$

$$\text{Mean value of } \frac{R_s}{R_A} = 3.583.$$

$$\text{Whence } \frac{R_s}{R_A L} = 1.801.$$

Thus the separate values of $\frac{R_s}{R_A L}$ are

$$1.800, 1.801, 1.798,$$

giving a mean value of 1.799.

The width of the specimen = 0.641 cm.

The thickness of the specimen = 0.0810 cm.

The specific resistance is given by the expression

$$\frac{R}{R_A L} \cdot \text{S.A.}$$

Thus the specific resistance ρ of "Dullray Soft" at 19.0° C.

$$= 1.799 \times 0.001 \times 0.641 \times 0.0810 \times 10^6 \text{ microhms-cm.}$$

$$= 93.01 \text{ microhms-cm. at } 19.0^\circ \text{ C.}$$

The following table (Table I.) summarizes the numerical results obtained for the specific resistances of the various

TABLE I.

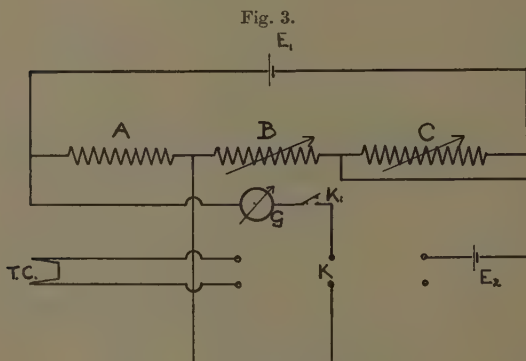
Substance.	$\frac{R_s}{R_A L}$	Mean $\frac{R_s}{R_A L}$	ρ microhms-cm.
Nickel soft	0.2106, 0.2105, 0.2105	0.2103	10.84
Nickel hard	0.2039, 0.2042, 0.2042	0.2040	10.57
Monel soft	0.8960, 0.8973, 0.8964	0.8966	46.8
Monel hard	0.9417, 0.9420, 0.9420	0.9419	48.7
48 per cent. nickel- iron soft	0.9842, 0.9835, 0.9845	0.9838	50.6
48 per cent. nickel- iron hard	1.026, 1.025, 1.022	1.023	52.5
Ferry soft	0.9640, 0.9640, 0.9651	0.9645	49.9
Ferry hard	0.9865, 0.9873, 0.9877	0.9875	50.8
Brightray soft . . .	2.106, 2.104, 2.104	2.105	109.6
Brightray hard . . .	1.914, 1.915, 1.912	1.913	99.0
Glowray soft	2.124, 2.127, 2.125	2.125	109.9
Glowray hard	1.995, 1.995, 1.995	1.995	103.5
Dullray soft	1.798, 1.800, 1.801	1.799	93.0
Dullray hard	1.766, 1.767, 1.766	1.766	91.6
20 per cent. nickel- silver soft	0.5968, 0.5970, 0.5975	0.5971	30.9
20 per cent. nickel- silver hard	0.5682, 0.5682, 0.5671	0.5678	29.5

alloys at 19.0° C. The three values of $\frac{R_s}{R_A L}$ given in the second column are the values obtained for the three different values of L used, as previously described, for each specimen.

IV. TEMPERATURE COEFFICIENT MEASUREMENTS.

(1) *Temperature Measurement.*

All temperature measurements were made with a base metal, chromel P-alumel thermocouple. The nominal composition of chromel P is 90 per cent. nickel, 10 per cent. chromium, whilst alumel contains approximately



95 per cent. nickel with aluminium, silicon, and manganese.

The "hot" junction of the thermocouple was formed by "spot welding" the two wires in a carbon arc. The cold junction was kept during calibration and subsequent temperature measurements in an ordinary thermos flask at 19.0° C. This cold junction temperature could be kept constant to within 0.1° C. for hours.

For calibration and temperature measurements the potentiometer circuit shown schematically in fig. 3 was used for the determination of the electromotive force of the thermocouple.

In this diagram E_1 represents a 2 volt accumulator, E_2 a standard cadmium cell, and T.C. the thermocouple.

A is a 1-ohm resistance provided with current and potential terminals, while B and C are calibrated resistance boxes each giving a maximum value of 1,100 ohms in steps of 0.05 ohm. By means of the double-pole double-throw switch K the electromotive force of the thermocouple is applied across A when the switch is in the left position; this electromotive force can then be balanced by variations of B and C, the galvanometer G serving to indicate balance. When balance has been obtained the e.m.f. of the thermocouple is numerically equal to the current through A, B, and C. To find the value of this current the switch K is placed in the right position, when the e.m.f. of the standard cell E_2 is applied across A and B in series; balance is now obtained by varying the ratio of B to C, their sum being maintained equal to its value in the previous case; from the values of B and C at balance and the value of the e.m.f. of the standard cell E_2 the current through A, B, and C, and therefore the e.m.f. of the thermocouple, can be determined.

Calibration was carried out at the following fixed points :—

- (i.) The boiling-point of water at a given pressure.
- (ii.) The melting-point of tin (A.R.) 231.9°C .
- (iii.) The melting-point of lead (A.R.) 327.3°C .
- (iv.) The melting-point of zinc (A.R.) 419.5°C .
- (v.) The melting-point of antimony (A.R.) 630.5°C .

For the steam-point a copper hypsometer was used, whilst for the metals cooling curves were obtained. The procedure consisted in heating the metals to about 10°C . higher than their melting-points; then they were allowed to cool slowly. Readings of the values of B and C for balance were noted every half minute, and from these readings resistance-time curves were plotted. From the value of the balancing resistance at the arrest point the e.m.f. of the thermocouple was calculated.

The melting was carried out in Salamander crucibles under a layer of graphite powder (to prevent oxidation); the thermocouple was protected from the molten metal with thin-walled pyrex tubing. Stirring of the melt was carried out with carbon rods, this, of necessity, being

696 Messrs. I. H. Thomas and R. M. Davies on the more vigorous in the case of tin and antimony, which had a tendency to supercool.

Results.

- (i.) E.M.F. per $^{\circ}\text{C.}$ between 19.0°C. and steam-point = 40.56 microvolts per $^{\circ}\text{C.}$
- (ii.) E.M.F. between 19°C. and 231.9°C. = 40.53 microvolts per $^{\circ}\text{C.}$
- (iii.) E.M.F. between 19°C. and 327.3°C. = 40.56 microvolts per $^{\circ}\text{C.}$
- (iv.) E.M.F. between 19°C. and 419.5°C. = 40.56 microvolts per $^{\circ}\text{C.}$
- (v.) E.M.F. between 19°C. and 630.5°C. = 40.53 microvolts per $^{\circ}\text{C.}$

Thus, within the limits of experimental error, the e.m.f. of the thermocouple is a linear function of the temperature.

Calibration was carried out several times during the course of this work; deviations from the above-quoted values were within the limits of experimental error at all times.

(2) The Electric Furnace.

The electric furnace consists of a silica tube 65 cm. long, 6 cm. internal diameter, and sealed at one end. The 30 cm. nearest the sealed end is coated with a thin layer of Alundum cement, on which the heating coil of nichrome wire (no. 22 S.W.G.) is wound. This winding consists of two sections of equal resistance, which may be used in series or parallel. The furnace resistance with the coils in series is 36 ohms, and with the coils in parallel 9 ohms.

The silica tube is mounted as shown in fig. 4, in which X and Y are two uralite disks holding the silica tube firmly in position in a sheet-iron cylinder CD. The diameter of CD is about 25 cm. and its length about 80 cm.; the open end of the silica tube protrudes about 10 cm. out of one end of CD, as shown. The space between the silica tube and the cylinder is tightly packed with Kieselguhr and asbestos flakes. This lagging is sufficient to make the furnace temperature independent of variations in room-temperature and of irregular

fluctuations in the 220 volt mains used to heat the furnace.

Fig. 4.

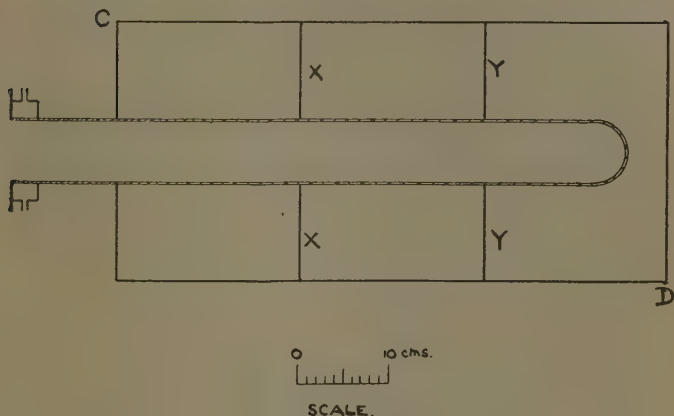
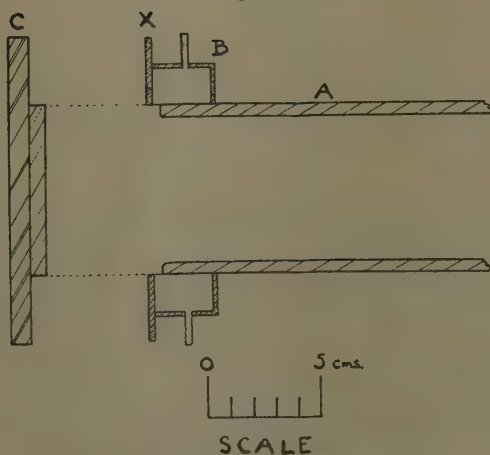


Fig. 5.

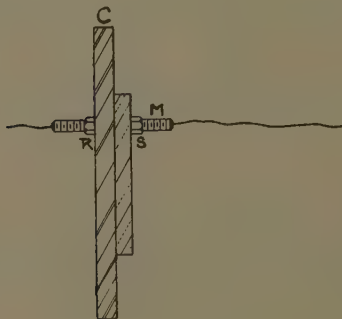


In order to avoid oxidation of the specimens at high temperatures it is necessary to be able to evacuate the silica tube, and for this purpose the open end is sealed as follows.

With reference to fig. 5, A represents the silica tube, whilst B is a cylindrical brass jacket to whose outer face a brass flange X is soldered. This jacket and flange fit tightly round the open end of the tube and are sealed in position with white wax. A stream of water is circulated through B in order to prevent the wax melting as the furnace temperature rises.

C is an uralite disk ground plane to the flange X; a hard rubber washer is placed between C and X, which are then firmly clamped together by six brass clamps arranged round the circumference. Four terminals are screwed through the cover C; these are used (as described below in Section IV. (3)) with the current and potential

Fig. 6.



leads. A wide glass tube, for evacuation purposes, also passes through C.

Some trouble was experienced with the sealing of the thermocouple wires through the uralite disk C. The first method tried consisted in drilling holes in C, passing the thermocouple wires through these holes, the wires being kept in position and sealing effected with white wax. This proved unsatisfactory, because the slightest movement of the wires cracked the wax, with the result that it proved impossible to evacuate the furnace.

This difficulty has been overcome in the following manner.

M is a small brass bolt drilled along its length. The thermocouple wire is then passed through this hole,

and is finally soldered to M, which is then fixed rigidly to the uralite disk with two nuts R and S, as shown, thus making an air-tight joint.

Thus in all C carries four terminals, a tube for evacuating, and two bolts carrying the thermocouple wires.

By using two thermocouples the temperature variation along the length and perpendicular to the length of the furnace has been investigated. With a uniformly spaced winding the temperature variation along the length of the tube is considerable, but by concentrating the winding at the two ends a length of about 10 cm. has been obtained, along which the maximum temperature variation is about 2°C . This compensation has been carried out at a temperature of approximately 300°C .; later experiments, however, showed that the furnace is under-compensated for temperatures less than 300°C . and over-compensated for temperatures greater than 300°C . The temperature variation along a cross-section of the tube is, however, negligible at all temperatures, and the apparatus for the determination of the temperature coefficient of resistance has thus been designed so that the material under test is placed across and not along the length of the furnace.

The above furnace was used in the investigation of the resistance-temperature relations of all the alloys with the exception of monel. This alloy has a Curie temperature in the neighbourhood of 100°C ., and at this temperature it is difficult and tedious to regulate the temperature of the furnace. For this reason the experiments on monel were carried out in an ordinary electrically heated oven; the temperature distribution inside this oven was found to be uniform in the space occupied by the specimen, and, instead of thermocouples, calibrated thermometers were used to measure the temperature of the specimen.

(3) The Measurement of the Temperature Variation of Resistance.

The specimens used in this measurement were about 6 cm. long, 0.6 cm. wide, and 0.05 cm. thick. For the reasons mentioned in Section IV. (2) the specimens had to be placed perpendicular to the length of the furnace,

and the method adopted for supporting the specimen and providing it with current and potential leads can be described with reference to figs. 7 and 7 (a); fig. 7 shows the apparatus in section and fig. 7 (a) in plan.

In these diagrams C represents the uralite furnace cover, which is fitted with four terminals. The two lower terminals serve as potential terminals, and to these brass clamps Y are screwed; these clamps are provided with slots in which thin flexible steel strips E about 60 cm. long, 2 cm. wide, and 0.2 cm. thick are fixed by means of screws. At a distance of about 40 cm. from C these

Fig. 7.

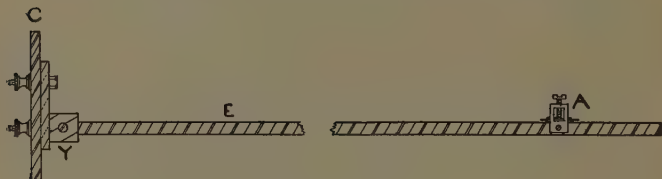
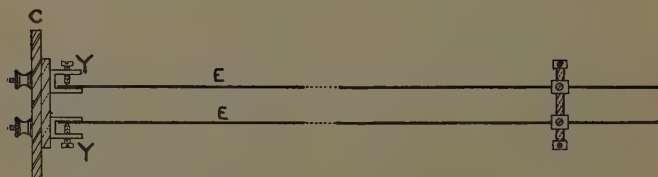


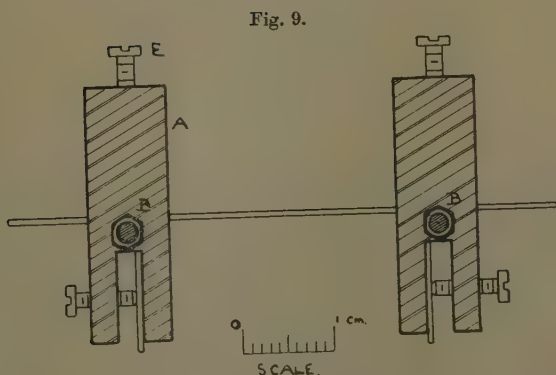
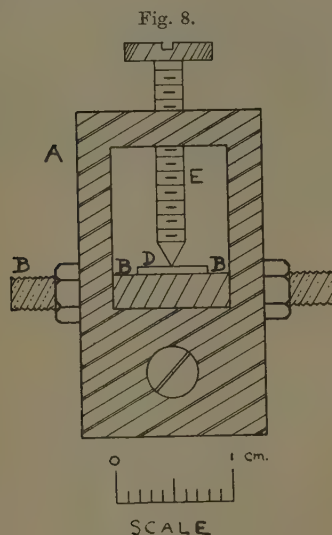
Fig. 7 (a).



strips carry potential terminals A, which serve to support the specimen. The construction of these terminals is shown in greater detail in figs. 8 and 9, fig. 8 showing an end section and fig. 9 a front section.

Referring to fig. 8, a potential terminal consists of a steel block A in which a rectangular hole has been milled; a constantan rod B, 2.6 mm. in diameter, rests on the lower surface of this hole, and passes through the sides, where it is clamped in position by nuts working on its screwed ends. The alloy specimen carries two such potential terminals as shown in fig. 9; the specimen is placed so that it rests on the rods B and so that the ends project about 1.0 cm.; the specimen is then clamped

firmly in position by means of the screws E. The rods B, on which the specimen is supported, are made of constantan because of the low thermal e.m.f. of this substance.

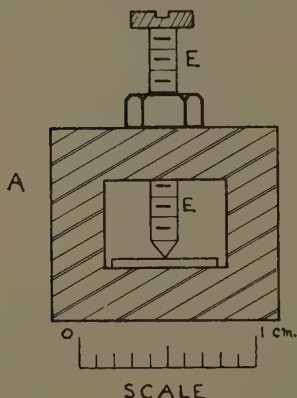


against most of the alloys examined. As shown in fig. 9 these terminals are provided with slots; the strips E of figs. 7 and 7 (a) fit tightly into these slots, and when

in position the strips are clamped by means of the screws shown in fig. 9. It is seen that the steel strips E fulfil two functions—they serve as leads between the specimen and the terminals on the furnace cover, and they also serve to support the specimen. In this last connexion it may be added that the strips are sufficiently flexible to allow unrestricted change in the length of the specimen on heating or cooling.

In addition to the two potential terminals the alloy specimen carries at each end a current terminal of the form shown in fig. 10.

Fig. 10.



The terminal consists of a steel block A, in which a hole slightly wider than the specimen has been milled; the terminal is slipped over the end of the specimen, which is then tightly clamped in position by means of the screw E. The two current terminals are connected to the two upper terminals on the furnace cover by means of thick copper wires insulated with beads.

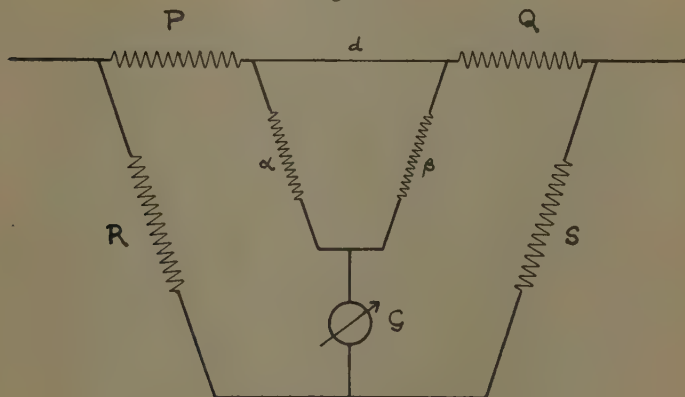
In the case of the measurements on monel in the electrically heated oven it was thought unnecessary to carry out the experiments in an evacuated space, since the highest temperature used in the experiments did not exceed 150°C .

The arrangement of the specimen together with the current and the potential leads was similar to that described above.

The "hot" junction of the thermocouple used for the measurement of the temperature of the specimen is allowed to rest gently on the specimen in the space between the potential terminals. This junction is insulated from the specimen and from the potential terminals by means of mica sheets.

In the course of some preliminary experiments the resistance measurements were carried out by means of a Kelvin double bridge, but it was found that, as far

Fig. 11.



as the present problem is concerned, this bridge method is unsatisfactory for two reasons.

To make this clear consider the bridge circuit represented by fig. 11, in which P denotes the unknown resistance, Q the standard resistance, d the resistance of the lead connecting the current terminals of P and Q; R, S, α , and β being resistance boxes. At balance

$$P = \frac{QR}{S} + \left(\frac{R}{S} - \frac{\alpha}{\beta} \right) \left(\frac{\beta d}{\alpha + \beta + d} \right).$$

Suppose that the alloy whose temperature coefficient of resistance is required is in the furnace and is fitted with current and potential terminals. When the specimen is being heated contamination will probably occur at the

potential terminals even though the furnace may be evacuated; this implies that an additional resistance will be included in series with the arms R and α , with the result that their values become indefinite and the results in consequence unreliable.

The second objection to the use of a Kelvin bridge in the present instance is the following. As we have seen it is necessary to use long leads to join the current terminals of the alloy and of the standard resistance; the bridge then becomes inaccurate ('Dictionary of Applied Physics,' vol. ii. p. 720), since the correction term

$$\left(\frac{\beta d}{\alpha + \beta + d}\right)\left(\frac{R}{S} - \frac{\alpha}{\beta}\right)$$

can become comparable with $\frac{QR}{S}$; moreover, this term

is indefinite, and varies with temperature because of the unknown temperature distribution along the portion of the leads within the furnace. A potentiometer method similar to that described in Section III. (1) does not suffer from these defects, since the contact resistance at the potential terminals of the specimen and the resistance of the leads joining the current terminals of the standard and unknown resistance do not affect the accuracy of the method. For these reasons the alloy resistance at different temperatures was determined by a potentiometer method in exactly the same manner as described under specific resistance determinations.

(4) *Accuracy of the Results.*

These determinations involve

- (i.) Temperature measurements,
- (ii.) Resistance measurements,

and are independent of the dimensions of the alloy under test.

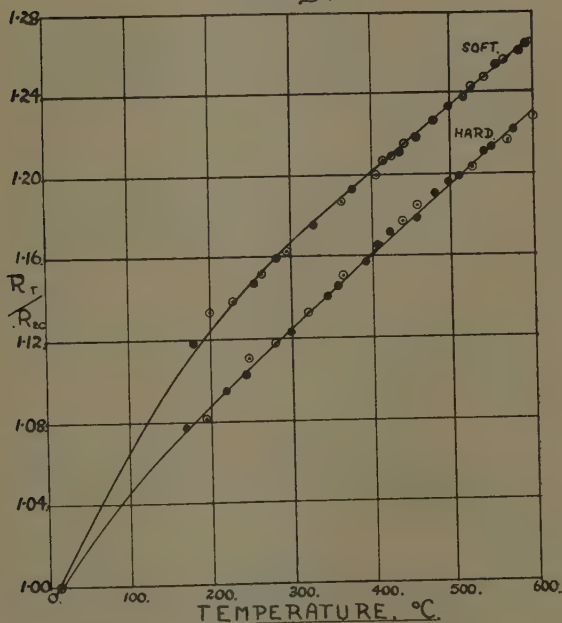
(i.) Taking into account the probable errors in the calibration of the thermocouple and subsequent temperature measurements any *steady* temperature could be determined to 0.1 per cent., and since the accuracy of the resistance boxes used in the determination is 0.1 per cent., temperature itself could be obtained to

0.1 per cent. However, the actual accuracy obtained is determined by the constancy of temperature during a resistance measurement.

(ii.) The resistance measurements were carried out as described in Section III. (1), where their accuracy has been discussed.

Since the quantity determined in these measurements

Fig. 12.

Dullray.

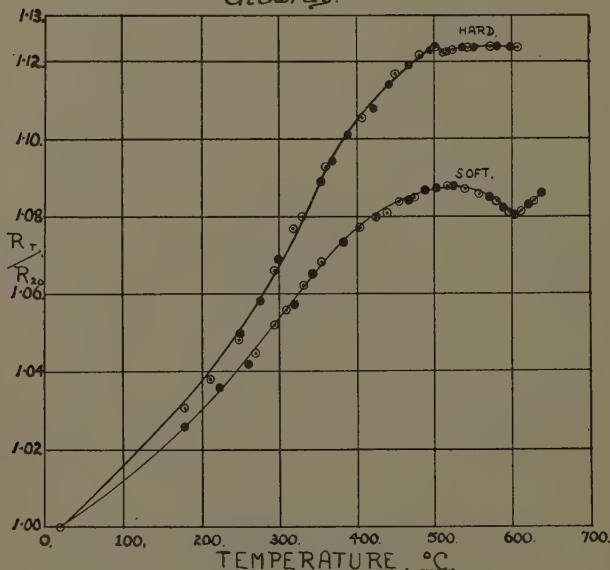
is the ratio of the resistance at a given temperature to the resistance at room-temperature, it follows that the dimensions of the specimen do not affect the accuracy of the results.

(5) Numerical Results for Temperature Variation.

The numerical results obtained are summarized in Table II., and are shown graphically in figs. 12-19.

In the figures the quantities plotted are the temperature T (in $^{\circ}\text{C}.$) and the ratio $\left(\frac{R_T}{R_{20}}\right)$, where R_{20} is the resistance of the specimen at $20^{\circ}\text{C}.$ and R_T its resistance at $T^{\circ}\text{C}.$ It will be seen that the highest temperature used was about $600^{\circ}\text{C}.$, which is well below the temperature at which the "soft" specimens were annealed. In the

Fig. 13.

Glowray.

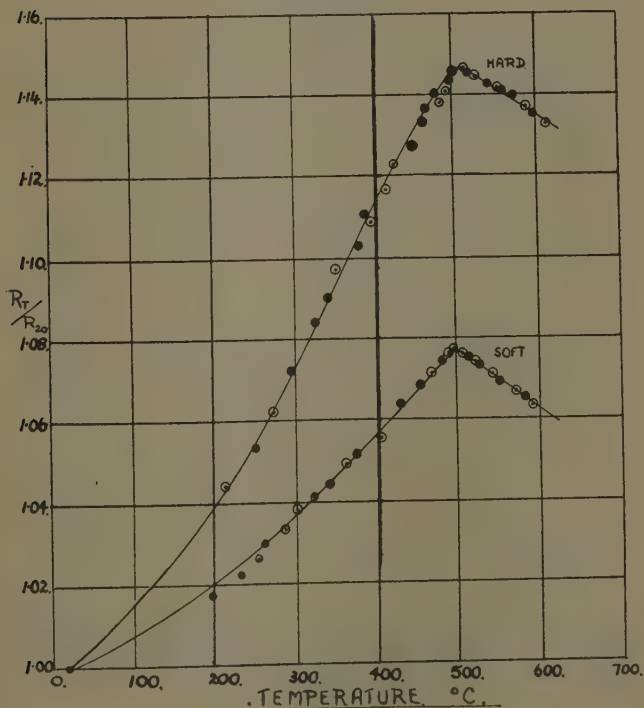
case of each alloy two "soft" specimens (cut from the same block) and two hard specimens (also cut from the same block) were investigated under conditions of temperature increasing and decreasing; in curves 12-18 the results for the first specimen are shown as points (●) and the results for the second specimen as points surrounded by circles (⊙).

The experimental values for the two specimens are seen to be in excellent agreement, apart from "ferry

soft" and "ferry hard." Here, however, it should be noted that the apparent discrepancy is due to the fact that, since the changes to be measured were so small, these curves have been plotted on a much greater scale than in the other diagrams.

Fig. 19 shows the results obtained for monel, the lower

Fig. 14.

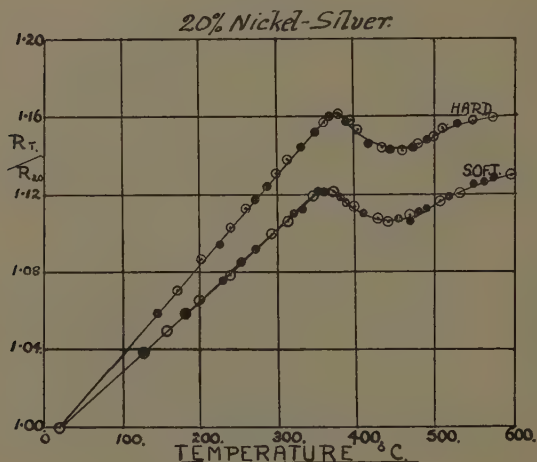
Bright-ray.

graph relating to the unannealed alloy and the upper graph to the annealed alloy.

In view of the discrepancies observed by Bates in the resistance-temperature curves obtained on successive heating and cooling of the ferromagnetic compound manganese arsenide (v. Section I.), it was decided to

carry out a similar series of experiments with monel. The results of these experiments are shown in the diagram as follows:—The circles with arrows ($\odot$) represent the points obtained on the first heating and cooling of the specimen, the points with arrows ($\bullet$) represent the results obtained on the second heating of the *same* specimen.

Fig. 15.



*Numerical Values of the Temperature Coefficient
of Resistance.*

The numerical values of the temperature coefficient of resistance $\alpha \left[\alpha = \frac{R_T - R_{20}}{R_{20}(T - 20)} \right]$ in the above notation

are given in the second column of Table II., the temperatures given in the third column being the upper and lower limits of the temperature for which α has been calculated: for purposes of comparison the values of the specific resistance of the alloys are given in the last column.

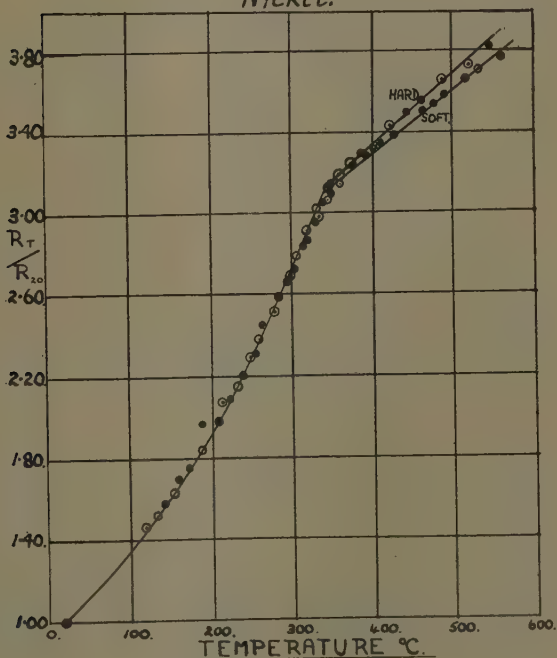
It should be noted that the temperature coefficient

of the specific resistance involves both α , and the coefficient of linear expansion β , of the alloys.

$$\frac{\rho_T}{\rho_{20}} = \frac{R_T}{R_{20}} [1 - \beta(T - 20)],$$

where ρ_T = the specific resistance at $T^\circ \text{C.}$, ρ_{20} = specific resistance at 20°C.

Fig. 16.

Nickel.

The temperature coefficient of the specific resistance (γ) will be given by

$$\gamma = \frac{\rho_T - \rho_{20}}{\rho_{20}(T - 20)} = \alpha - \frac{R_T}{R_{20}} \cdot \beta.$$

The relative magnitudes of α and γ may be seen in the

Fig. 17.

48% Nickel-iron.

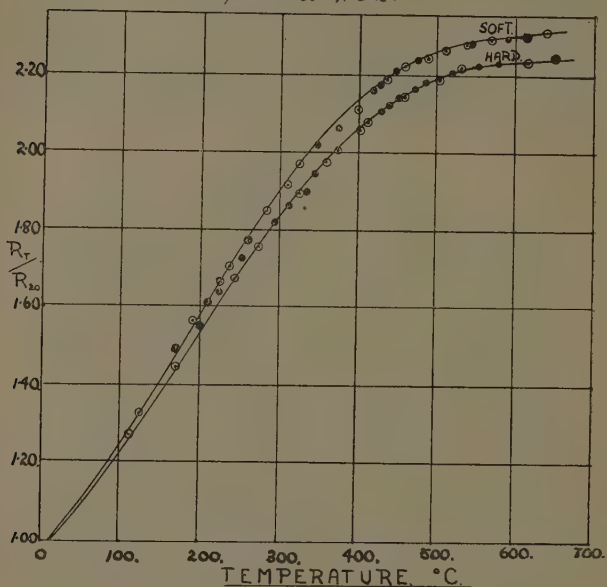


Fig. 18.

Ferry.

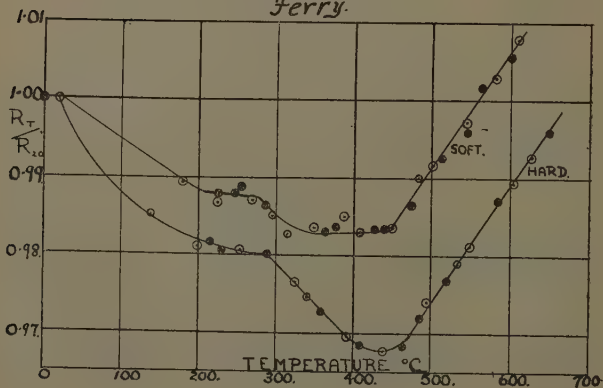


Fig. 19.

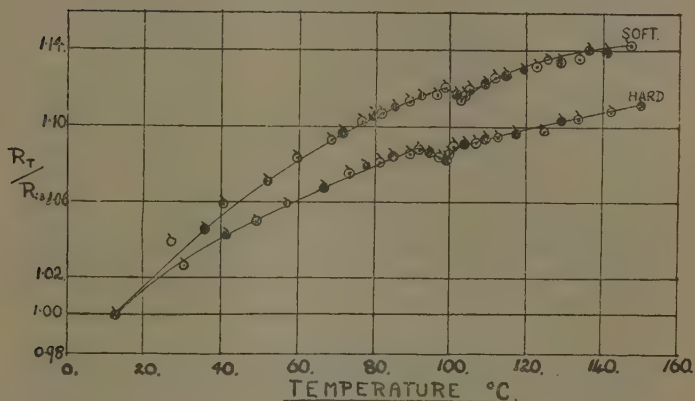
Monel.

TABLE II.

Substance.	Temperature coefficient of resistance.	Temperature range.	Specific resistance (ohm-cm. $\times 10^{-6}$).
Nickel hard	0.00648	20 $^{\circ}$ C.-350 $^{\circ}$ C.	10.57
Nickel soft	0.00655	20 $^{\circ}$ C.-350 $^{\circ}$ C.	10.84
20 per cent. nickel-silver hard	0.000456	20 $^{\circ}$ C.-500 $^{\circ}$ C.	29.5
20 per cent. nickel-silver soft	0.000350	20 $^{\circ}$ C.-500 $^{\circ}$ C.	30.9
Brightray hard ...	0.000302	20 $^{\circ}$ C.-500 $^{\circ}$ C.	99.0
Brightray soft ...	0.000158	20 $^{\circ}$ C.-500 $^{\circ}$ C.	109.6
Glowray hard	0.000256	20 $^{\circ}$ C.-500 $^{\circ}$ C.	103.5
Glowray soft	0.000181	20 $^{\circ}$ C.-500 $^{\circ}$ C.	109.9
Dullray hard	0.000406	20 $^{\circ}$ C.-500 $^{\circ}$ C.	91.6
Dullray soft	0.000492	20 $^{\circ}$ C.-500 $^{\circ}$ C.	93.0
Ferry hard	0.0000825	20 $^{\circ}$ C.-425 $^{\circ}$ C.	50.8
Ferry soft	0.0000433	20 $^{\circ}$ C.-425 $^{\circ}$ C.	49.9
Monel hard	0.00103	13 $^{\circ}$ C.-100 $^{\circ}$ C.	48.7
Monel soft	0.00138	13 $^{\circ}$ C.-100 $^{\circ}$ C.	46.8
48 per cent. nickel-iron hard	0.00248	20 $^{\circ}$ C.-500 $^{\circ}$ C.	52.5
48 per cent. nickel-iron soft	0.00262	20 $^{\circ}$ C.-500 $^{\circ}$ C.	50.6

case of nickel soft, where, for the temperature range 20°C. – 350°C. ,

$$\alpha = 655 \times 10^{-5},$$

$$\frac{R_T}{R_{20}} = 3.1,$$

$$\beta = 1.3 \times 10^{-5},$$

and

$$\frac{R_T}{R_{20}} \cdot \beta = 4 \times 10^{-5}.$$

The difference between α and γ thus amounts to about 0.6 per cent. in this instance. Unfortunately, owing to absence of data for β , it has not been possible to calculate γ for the alloys.

Results of Previous Investigations on the Temperature Coefficient of Resistance of Nickel and some of the Alloys.

Table III. gives the values found in the literature for the temperature coefficient of specific resistance of nickel and the temperature coefficient of resistance of two of the alloys, together with the temperature range for which this quantity was determined. The second column indicates whether the measurements were carried out on annealed or unannealed specimens.

TABLE III.

Substance.	Annealed or unannealed.	Temperature range.	Temperature coefficient.	Reference.
Nickel	A.	19°C. – 350°C.	0.00658	Schofield *.
	A.	19°C. – 350°C.	0.00651	Present work.
	U.	19°C. – 350°C.	0.00644	Present work.
Glowray ..	?	0°C. – 200°C.	0.00030	Rohn †.
	A.	0°C. – 200°C.	0.00018	Present work.
	U.	0°C. – 200°C.	0.00025	Present work.
Brightray .	?	0°C. – 200°C.	0.00010	Rohn.
	A.	0°C. – 200°C.	0.00015	Present work.
	U.	0°C. – 200°C.	0.00030	Present work.

* Schofield, Proc. Roy. Soc. (A) cvii. p. 206 (1925).

† Rohn, *Electro-Tech. Zeit.* xlviii. p. 227 (1927).

V. DISCUSSION OF THE RESULTS.

- (1) *The Relation between the Specific Resistance and Temperature Coefficient of Resistance of the Alloys in the Annealed and Unannealed States.*

Examination of the results given in Section III. (2) shows that the alloys can be divided into two categories as regards the relation between the specific resistance in the annealed and unannealed states. For the first class the specific resistance of the annealed specimens is greater than that of the unannealed specimens; to this class belong nickel, 20 per cent. nickel-silver, brightray, glowray, and dullray. For the second class the specific resistance of the annealed specimens is less than the unannealed specimens; to this class belong 48 per cent. nickel-iron, ferry, and monel.

It is difficult to draw any theoretical conclusions from these results except those stated in Section I. as to the magnitudes and signs of ρ_A (the resistance due to disorder) and the coefficients K_0' and K_1' of equations (i.) and (ii.).

In all cases except ferry the temperature coefficient of resistance (in the neighbourhood of room-temperature) is positive.

The substances investigated can also be divided into two categories as regards the temperature coefficients of the annealed and unannealed specimens. The first category comprises those substances for which the temperature coefficient is greater when in the annealed state; to this class belong nickel, dullray, 48 per cent. nickel-iron, and monel. The second category consists of those substances whose temperature coefficients are greater in the unannealed state; to this class belong 20 per cent. nickel-silver, brightray, glowray, and ferry. Inspection of the curves showing the temperature variation of R_T/R_{20} shows that the general character of the curves is the same for annealed and unannealed specimens.

- (2) *The Temperature Variation of the Resistance of Ferromagnetic Substances in the neighbourhood of the Curie Temperature.*

The ferromagnetic substances investigated are glow-ray, nickel, and 48 per cent. nickel-iron and monel. The former is a ternary alloy, and its Curie temperature and

phase diagram do not appear to be given in the literature. The R_T/R_{20} temperature curves (fig. 13) for the annealed and unannealed specimens show a gradual increase of R_T/R_{20} from room-temperature to a maximum; this is followed by a slight decrease to a minimum, and this in turn is followed by a progressive increase. In the case of the unannealed specimen the maximum occurs at about 500°C ., and in the case of the annealed specimen at about 515°C . The minimum occurs at about 510°C . for the unannealed specimen and at about 600°C . for the annealed specimen. It is interesting to note that these discontinuities occur at different temperatures for the annealed and unannealed specimens; there is also a slight difference in the character of the curves for the two specimens.

As regards nickel (v. fig. 16) the experimental curves for the hard and "soft" specimens coincide below the Curie temperature, but above this temperature the increase of resistance of the hard specimen is greater than that of the "soft" specimens. It is also clear that there is a pronounced discontinuity in the R_T/R_{20} -temperature curves at 350°C . (the Curie temperature). Above this temperature the R_T/R_{20} -temperature relation is linear. This result is in agreement with the results of the previous investigations of Gerlach u. Schneiderhan⁽⁴⁾ and Schofield⁽⁷⁾. It is also seen that the general character of the curves is the same for the annealed and unannealed specimens.

Next, considering the case of 48 per cent. nickel-iron (fig. 17), the R_T/R_{20} -temperature curve gives no indication of a discontinuity even in the neighbourhood of the Curie temperature (425°C .). This behaviour should be contrasted with that of nickel above and other substances such as manganese arsenide (v. Section I.), where pronounced discontinuities occur at the Curie temperature. On theoretical grounds it is difficult to see how the transition through the Curie temperature can occur without some change in the form of the R_T/R_{20} -temperature curve.

Finally, considering the case of monel, quite definite discontinuities occur at approximately 100°C ., which must be correlated with the magnetic change of state at this temperature. It should be noted that the temperature at which this discontinuity occurs is lower

for the unannealed alloy than for the annealed alloy. It is seen from the graph that both sets of points lie on a smooth curve for both annealed and unannealed specimens. This is also true of the results obtained on subsequent heating of these specimens (points not recorded in fig. 19), and also of the results obtained with other specimens of this alloy when taken through similar processes of heating and cooling.

(3) *The Temperature Variation of the Resistance of Non-magnetic Substances.*

The R_T/R_{20} -temperature curves for dullray (fig. 12) require no comment. In the case of brightray (fig. 14) the R_T/R_{20} -temperature curves show a progressive increase of resistance from room-temperature up to a temperature in the neighbourhood of 500°C. , where a pronounced maximum appears in the curves; for temperatures above 500°C. the resistance decreases with temperature. The phase diagram of this substance indicates that no structural change occurs at this temperature, and the behaviour of this substance is thus similar to that of a number of other alloys (*v.* 'Dictionary of Applied Physics,' vol. v. p. 260). The case of 20 per cent. nickel-silver (*v.* fig. 15) is somewhat similar, except that the maximum at 375°C. is followed by a minimum in the neighbourhood of 450°C. ; beyond the latter temperature the resistance again increases with temperature.

The experiments with ferry (fig. 18) were difficult owing to the small changes involved; consequently it is not easy to make any definite statement concerning the variation of resistance of ferry with temperature. There is indication, however, of a flattening of the R_T/R_{20} -temperature curve in the neighbourhood of 250°C. , and there is certainly a minimum in the neighbourhood of 425°C.

In conclusion, it has to be admitted that, from the *theoretical standpoint*, the results of the present work are somewhat disappointing in so far as it seems impossible to correlate with theory our experimental results, which, so far as we know, are new. In connexion with the relation between the specific resistances of the annealed and unannealed alloys, there appears to be no definite

rule; approximately half of the alloys have a greater specific resistance in the annealed state, whilst for the remainder the converse is true. In addition, the anomalous behaviour of 48 per cent. nickel-iron appears to show that the existence of a Curie point is no certain criterion of the occurrence of a discontinuity in the resistance-temperature curve. Furthermore, the behaviour of the non-magnetic alloys, ferry, brightray, and 20 per cent. nickel-silver indicates that discontinuities in the resistance-temperature curve can occur without any apparent structural change so far as is indicated by the phase diagrams for these alloys. The results indicate that possibly more definite theoretical conclusions could be drawn from the results of a systematic investigation of a binary alloy system, such as the nickel-iron system, varying progressively the concentration of the two components.

We desire to place on record our sincere thanks to Vice-Principal Gwilym Owen for his unfailing support and encouragement in many directions, and to Dr. Emlyn Stephens, Messrs. L. E. Sulston and D. Owen for valuable suggestions both theoretical and practical. In addition, our best thanks are due to Messrs. Henry Wiggin, Ltd., for their gift of the alloys. One of us (I. H. T.) wishes to thank the Department of Scientific and Industrial Research for a maintenance grant.

Note added in proof, May 19th, 1936.

Schulze * has recently shown experimentally that in the case of annealed specimens of iron and nickel there exists a minor discontinuity in the resistance-temperature curves in addition to the major discontinuity at the Curie point. This minor discontinuity occurs with only annealed specimens and it occurs at a temperature below the Curie point; in the case of nickel, it occurs at about 220° C., the Curie point being about 350° C.

It is interesting to note that the curve for nickel (fig. 16) shows two points at temperatures in the neighbourhood of 200° C. which do not lie on the smoothed curve; these two points relate to the annealed specimens,

* *Zeit. für Metallkunde*, xxvii. p. 251 (1935). *Phys. Zeit.* xxxvii. p. 41 (1936).

whilst the points lying on the curve relate to the unannealed specimens. The experimental error is less than the deviation of these points, and this deviation is therefore probably due to the effect recorded by Schulze. It is also seen from the curves for glowray, 48 per cent. nickel-iron, and monel (figs. 13, 17, 19) that the effect does not appear to be shown by these ferromagnetic substances.

VI. BIBLIOGRAPHY.

- (1) Tammann, *Z. Anorg. Chem.* cvii. p. 1 (1919).
 - (2) Johansson and Linde, *Ann. d. Physik*, lxxviii. p. 439 (1925).
 - (3) Bradley and Jay, *Proc. Roy. Soc. (A)* cxxxvi. p. 210 (1932).
 - (4) Gerlach. u. Schneiderhan, *Ann. d. Physik*, vi. p. 772 (1930).
 - (5) Bates, *Phil. Mag.* xvii. p. 783 (1934).
 - (6) Weaver, *Mitt. Kais. Inst. Eisenforsch. Dusseld.* ix. p. 151 (1927).
 - (7) Schofield, *Proc. Roy. Soc. (A)* cvii. p. 206 (1925).
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LV. *Magnetic Birefringence in Solutions of Organic Substances.* By L. D. MAHAJAN, M.Sc., Professor of Physics, Mohindra College, Patiala *.

1. Introduction.

IN a recent paper † Dr. S. W. Chinchalkar has studied the magnetic birefringence of a number of organic compounds containing two or more benzene rings, in solution in suitable solvents. From the magnitude of the birefringence he has drawn some important conclusions regarding the structure of the molecules, and the influence of the structure on the magnetic and the optical anisotropies of the molecules. For example, he finds that in diphenyl, diphenylbenzene, triphenylbenzene, etc., the different benzene rings in the molecule lie either in the same plane or in parallel planes. "Condensation" of the benzene rings, as in naphthalene, or anthracene, or phenanthrene, is found to increase both the magnetic and the optical anisotropies.

The present paper concerns itself with similar measurements on a number of simple derivatives of benzene, naphthalene, and anthracene, and an interpretation of the results from the same point of view.

* Communicated by Prof. K. S. Krishnan, D.Sc.

† In course of publication in *Proc. Ind. Acad. Sc.*

2. *Experimental Method.*

The experimental arrangement used for the measurement of magnetic birefringence was the same as that adopted by Chinchalkar. The birefringence of the solution was measured with the help of a Rayleigh bent glass compensator, by comparing the shift of the dark band in the compensator with the shift produced, under identical conditions, by a liquid of known magnetic birefringence. Either acetone or benzene was used for this purpose, the former when the birefringence to be measured was small, and the latter when it was large.

Carbon tetrachloride, being inactive, was selected as the solvent whenever convenient. But several of the substances were not soluble to any appreciable extent in this liquid, and for them, acetone, or acetic ether, or chloroform, which exhibited only a feeble magnetic birefringence, was used as the solvent. For these solutions the contribution from the solvent to the observed birefringence, has to be calculated and allowed for.

Among the substances studied, durene, hexaethylbenzene, ortho-toluidine, and phenanthraquinone were from Kodak; quinone, anthraquinone, catechol, and resorcinol from Kahlbaum; hydroquinone, *m*-nitroaniline, α -naphthoquinone, and α -naphthylamine from Merck; chloranil, bromanil, phloro-glucinol, benzene hexachloride, and diphenylene oxide from Schuchardt; and hexachlorobenzene from British Drug House. The substances were, in general, purified by recrystallization or sublimation; sometimes, as with quinone, the purification was done just before use.

Some of the solutions showed a tendency to be colloidal, and particular care was taken to guard against it (since colloidal particles often gave abnormally high birefringence), by either changing the solvent or by suitably altering the conditions under which the solutions were prepared. For example, solutions of durene, hexamethylbenzene, and of α -naphthylamine in carbon tetrachloride, which showed this tendency, exhibited high negative birefringence; those of dimesityl, and α -naphthoquinol in acetic ether gave similarly a very high positive birefringence, as also hexachlorobenzene in chloroform.

Some of the solutions slightly decomposed under the action of the incident light. For example, catechol, resorcinol, and hydroquinol turned red under the action

of light, while catechol gave a dark precipitate. For such solutions measurements were taken with the minimum of exposure necessary.

Many of the solutions were strongly coloured, as, for example, the solutions of the quinones, and they had, therefore, to be used in very low concentrations.

3. Results.

The results obtained are given in the table. The measurements were all made at room-temperature, which varied from 28° to 31° C.

The values of the birefringence C_m of the solution (defined by the usual relation,

$$C_m = \frac{n_{\parallel} - n_{\perp}}{\lambda H^2},$$

where $n_{\parallel}$ and $n_{\perp}$ are the refractive indices for light-vibrations parallel and perpendicular respectively to the magnetic field, λ is the wave-length of the light, and H is the strength of the magnetic field), are given in column 6 of the table. For convenience the birefringence values are expressed in terms of the usual unit, in which the birefringence of liquid nitrobenzene is taken to be 100. Column 7 gives the contribution from the solvent to the observed birefringence of the solution, and column 8 the contribution from the solute, obtained by subtracting the value given in column 7 from that in column 6. Dividing the figures in column 8 by the gram-molecular concentration, we obtain the birefringence C_M of the substance *per gram molecule per c.c.* The values of C_M are given in column 9.

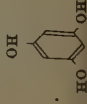
For benzene liquid in the same unit ($C_m^{\text{nitrobenzene}} = 100$), $C_m = 23.3$, and taking its density as 0.88 and molecular weight = 78, C_M^{benzene} comes out as $\frac{23.3 \times 78}{.88} = 20.7 \times 10^2$.

The values of C_M for the various substances, divided by C_M^{benzene} , are given in the last column of the table.

4. Discussion.

(a) *Benzene Derivatives.*—For most of the benzene derivatives, the gram molecular birefringence is nearly of the same magnitude as for benzene, C_M/C_M^{benzene} ranging from about 1 to 1.7, suggesting that the bulk of both the magnetic and the optical anisotropies of the

TABLE OF RESULTS.

Serial No.	Substance.	Solvent.	Concentration of solution		Magnetic birefringence relative to that of nitrobenzene=100				$\frac{C_M}{C_{\text{benzene}}}$
			in gms. per c.c. $\times 10^2$,	in gm. mols. per c.c. $\times 10^4$,	of solution.	contributed by solvent.	of the substance in solution.	per gm. mol. of substance in per c.c. $\times 10^{-2}$.	
1	Catechol (<i>o</i> -dihydroxybenzene). 	Acetic ether	4.37	3.97	1.93	0.70	1.23	31	1.5
2	Resorcinol (<i>m</i> -dihydroxybenzene). 	"	1.80	1.63	1.20	0.71	0.49	30	1.5
3	Hydroquinol (<i>p</i> -dihydroxybenzene). 	"	3.42	3.11	1.60	0.70	0.90	29	1.4
4	Phloroglucinol (1, 3, 5-trihydroxybenzene). 	Acetone	2.79	2.21	2.36	1.58	0.78	35	1.7

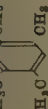
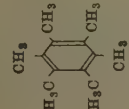
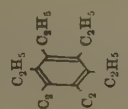
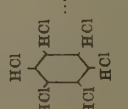
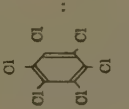
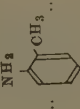
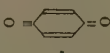
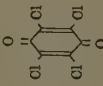
5	 { Durene (<i>s</i> -tetramethyl- benzene). 	Acetone	0.71	0.53	1.79	1.60	0.19	36	1.7
6	 { Hexamethyl- benzene. 	"	1.84	1.13	1.96	1.57	0.39	35	1.7
7	 { Hexaethyl- benzene. 	Carbon tetra- chloride	6.72	2.73	0.81	0.00	0.81	30	1.5
8	 { Benzene hexa- chloride. 	Acetic ether	8.81	3.03	0.71	0.68	{ 0.03 } (negli- gible)	1	{ .05 } (negli- gible)
9	 { Hexachloro- benzene. 	Carbon bi- sulphide	4.07	1.43	-19.07	-19.55	0.48	34	1.6
10	 { Ortho-toluidine 	Acetone	3.79	3.54	2.45	1.56	0.89	25	1.2

TABLE OF RESULTS (cont.).

Serial No.	Substance.	Solvent.	Concentration of solution		Magnetic birefringence relative to that of nitrobenzene = 100				$\frac{C_M}{C_{\text{benzene}}}$
			in gms. per c.c. $\times 10^2$.	in gm. mols. per c.c. $\times 10^4$.	of solution.	contributed by solvent.	of the substance in solution.	per gm. mol. of substance per c.c. $\times 10^{-2}$.	
11	<i>m</i> -Nitroaniline	Acetic ether	0.92	0.67	1.49	0.71	0.78	116	5.6
									
12	<i>p</i> -Benzoquinone	Acetone	0.52	0.48	2.02	1.60	0.42	87	4.2
									
13	Chloranil	"	1.20	0.49	2.03	1.60	0.43	88	4.3
									
14	Bromanil	"	1.16	0.27	1.84	1.59	0.25	93	4.5
									

15	Dimesityl		tetra- chloride	0.97	0.41	0.42	0.00	0.42	102	4.9
16	Diphenylene oxide.		"	3.05	1.81	2.66	0.00	2.66	147	7.1
17	α -Naphthylamine		Acetone	3.91	2.33	5.32	1.55	3.77	162	7.8
18	α -Naphthoquinone		"	1.13	0.79	2.62	1.59	1.03	130	6.3
19	Anthraquinone		Chloroform	0.64	0.41	-2.09	-2.72	0.63	154	7.4
20	Phenanthra- quinone.		"	0.80	0.38	-1.28	-2.73	1.45	382	18.5
21	Benzene		Acetone	0.93	0.45	2.87	1.59	1.28	284	13.7
				88	113	23.3	—	23.30	20.7	{ 1 (used as a stan- dard)

molecule is due to the benzene ring in it. In benzoquinone, however, the gram molecular birefringence is much higher, being nearly four times that of benzene. This is easily understandable from the structure of the benzoquinone molecule, $\text{O}=\text{C}_6\text{H}_4=\text{O}$, which has recently been analysed by X-rays by Robertson *. The optical polarizability of the molecule along the line joining the two O-atoms should be considerably larger than that of benzene for directions in its plane, and this factor alone should explain much of the observed increase in the birefringence as compared with that of benzene. The birefringence of its two derivatives, chloranil and bromanil, is also of the same order of magnitude, as we should expect.

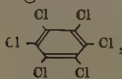
Hydroquinol gives of course normal birefringence, since the structure of the benzene ring in this compound, $\text{OH}-\text{C}_6\text{H}_4-\text{OH}$, is the same as in benzene, and very different from the ring in benzoquinone.

NH_2 -derivatives of benzene have very nearly the same birefringence as benzene, for example the birefringence of

ortho-toluidine, $\text{CH}_3-\text{C}_6\text{H}_4-\text{NH}_2$, is 1.2 times that of benzene, and is comparable with that of aniline, $\text{C}_6\text{H}_5-\text{NH}_2$. It is smaller than that of the hydroxy-derivatives.

The gram molecular birefringence of *m*-nitroaniline, containing one NO_2 group, is about 5.6 times that of benzene, which is very high, and is comparable with that of nitrobenzene (five times that of benzene). The high value is due to the presence of the NO_2 group, which is known to increase the birefringence enormously.

The striking contrast in the optical and magnetic anisotropies of benzene and cyclohexane is well known †. The same contrast is brought out in the corresponding hexachloro-derivatives; whereas hexachlorobenzene,



exhibits a strong birefringence, benzenehexa-



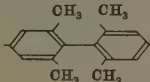
chloride, $\text{HCl}-\text{C}_6\text{H}_6-\text{HCl}$, is almost inactive.



* Proc. Roy. Soc. A, cl. p. 106 (1935).

† Raman and Krishnan, Proc. Roy. Soc. A, cxiii. p. 511 (1927).

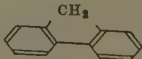
(b) *Diphenyl Derivatives*.—Two diphenyl derivatives were studied, viz., dimesityl and diphenylene oxide. The gram

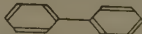
molecular birefringence of dimesityl, H_3C  CH_3 ,

is nearly five times that of benzene, and it suggests that, as in the case of diphenyl and its other derivatives studied by Chinchalkar, in this molecule also the two benzene rings are nearly coplanar.

The birefringence of diphenylene oxide, , is

however, considerably higher (about 7.5 times that of ben-

zene), being much nearer to that of fluorene, 

(eight times), than to that of diphenyl, 

(four times). This again should be expected from the structures of the molecules, since both diphenylene oxide and fluorene contain an extra closed ring which is not present in diphenyl.

(c) *Molecules with condensed Benzene Rings*.—All the compounds containing condensed benzene rings, given at the end of the table, have very high birefringence. α -naphthylamine has nearly the same birefringence as naphthalene studied by Chinchalkar, while naphthoquinone gives naturally a slightly higher value.

Similarly, anthraquinone and phenanthraquinone have a slightly higher birefringence than the corresponding hydrocarbons.

The author takes this opportunity to thank Prof. K. S. Krishnan for suggesting the problem, and constant guidance during the progress of this work. The author also thanks Dr. S. W. Chinchalkar, who helped the author in the progress of the work.

LVI. *The Calculation of Surface Tension from Experiment.* By ALFRED W. PORTER, D.Sc., F.R.S., Emeritus Professor in the University of London*.

PART IV.

THE rise, h_0 , in tubes of circular section, of various widths, for angles of contact between 0° and 90° .

This paper takes up the same question dealt with in Part II. [Phil. Mag. 7, xxi. p. 703 (1936)] for zero angle of contact and extends it to various angles. This extension has been made with the aid of Bashforth and Adams's Tables, the variables being transformed to suit experimental conditions. In fig. 1 are shown the values of Δ_α in the general formula

$$\frac{\beta^2}{r^2} \cos \alpha = \frac{1}{2} \frac{h_0}{r} + \Delta_\alpha,$$

where r = radius of tube, h_0 = capillary elevation, β^2 = capillary constant = $\sigma/g(\rho_{\text{liq}} - \rho_{\text{vap}})$ †.

The abscissae represent the experimentally determined values of $\frac{1}{2} \frac{h_0}{r}$. The curves shown are for values of α equal to 0° , 15° , 30° , 45° , 60° , and 75° . The steeper curves which are drawn across them correspond to constant values of $\frac{h_0}{\beta}$; these are calculated in the first place directly from the tables for various acute contact angles, and the points corresponding to a given contact angle are then connected together to constitute the final curves of Δ_α corresponding to that angle. The solid dots on some of the constant $\frac{h_0}{\beta}$ curves serve to divide the 15 degree intervals into intervals of 5 degrees.

* Communicated by the Author.

† It might seem that a better way would have been to have calculated

$$\frac{\beta^2}{r^2} = \frac{1}{2 \cos \alpha} \frac{h_0}{r} + \text{correction.}$$

But the correction-curves so obtained intersect one another and the diagram obtained is very confusing.

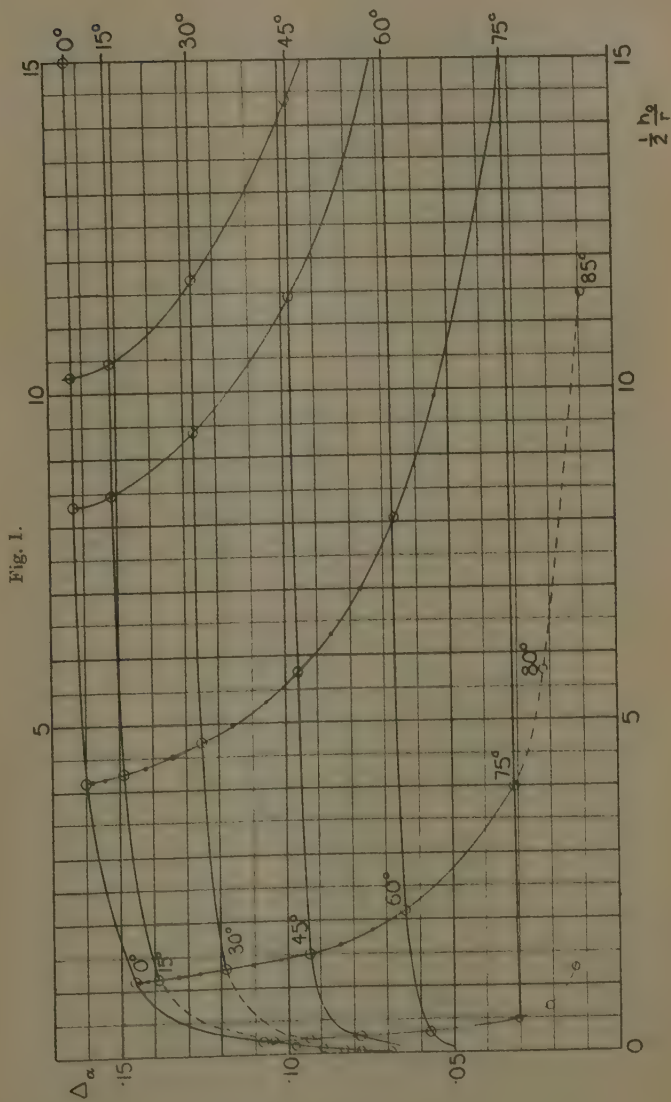
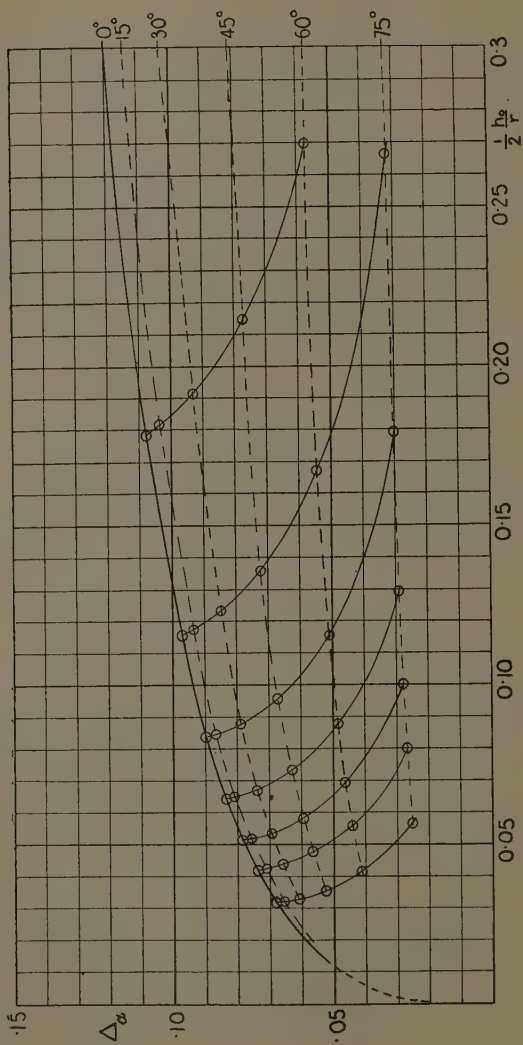


Fig. 2.



The great range of $\Delta\alpha$ to be dealt with makes it impracticable to represent the whole data on one diagram and scale. It will be observed that while $\Delta\alpha$ is fairly constant for a given angle, α , towards the right-hand of fig. 1, yet it becomes very steep on the left-hand, where h_0/r is small; in consequence fig. 1 cannot be used for abscissæ smaller than 0.3. For smaller abscissæ recourse must be had to fig. 2, which is on a much more extended horizontal scale. These two figures give curves for practically the whole range of data calculable from the tables.

When, as in the case of mercury, depression (instead of rise) occurs, the tables still apply if α is then taken as the *acute* angle of contact and h_0 is the experimental value of the *depression*.

For cases in which the abscissæ are larger than those of fig. 1 the need for tables becomes negligible, because the greatest value of $\Delta\alpha$ is $1/6$, and its value can then be obtained with sufficient accuracy by the well-known approximate formulæ.

The accompanying table gives the values calculated in various cases from figs. 1 and 2 by way of illustration of the use of the charts.

$\frac{1}{2} \frac{h_0}{r}$	Acute contact angle.	$\Delta\alpha$.	$\frac{\beta^3}{r^2} \cos \alpha$.	$\frac{\beta^2}{r^3}$.
15	0°	·164	15·164	15·164
5	0°	·161	5·161	5·161
5	15°	·150	5·150	5·232
5	30°	·126	5·126	5·919
0·2	0°	·111	·311	0·311
·2	15°	·106	·306	·317
·2	30°	·094	·294	·339
·1	0	·094	·194	·194
·1	30°	·082	·182	·205
·05	0°	·075	·125	·125

These figures should be compared with those often given for the "capillary correction."

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Phil. Mag. S. 7. Vol. 22. No. 148. Oct. 1936.

3 B

LVII. *Viscous Motion under Gravity in a Liquid Film adhering to a Vertical Plate.* By GEORGE GREEN, D.Sc., *Lecturer in Physics in the Applied Physics Department, The University, Glasgow* *.

THE problems which present themselves for solution in connexion with the draining of a vertical plate have been treated mainly from the experimental standpoint by Satterly and others †. A theoretical discussion of the fundamental hydrodynamical problem involved has been given by Jeffreys ‡, who, however, employs the assumption that the motion at any point within the film adhering to the vertical plate may be regarded as practically a steady fluid motion. On that basis a solution representing the profile of the film in the later stages of the motion is obtained. In the present paper an attempt is made to take into account the initial motion within the film and so determine its complete progress starting from definite assumed initial conditions. It is hoped that the more complete theoretical treatment of the problem here given may lead to the removal of discrepancies hitherto found between theory and experiment.

Consider a mass of liquid, initially at rest, bounded by a fixed vertical plate at the plane $x=0$, and extending to practically infinite distance in the x direction. At instant $t=0$ let the liquid be lowered from the plate; that is, we suppose that the liquid is left free to fall under gravity retarded only by its own viscosity and by its contact with the vertical plate. In this manner at any later instant a film of liquid is obtained in contact with the vertical plate extending from the original surface level to the new level of the main body of the liquid. On the exposed surface of this film the atmosphere exerts a uniform pressure, and over the horizontal surface cutting off the film from the lowered liquid the pressure gradient in a horizontal direction is zero. Hence, in accordance with the usual theory of motion in a boundary layer, we make the assumption that the pressure throughout

* Communicated by the Author.

† See Trans. Roy. Soc. Can. xxvii. (1933), which contains additional references.

‡ Proc. Cam. Phil. Soc. xxvi. p. 204 (1929-30).

the whole film is sensibly constant. It is probable that there is very little taken for granted in assuming also that within the film the motion is completely laminar, with each fluid particle descending in a vertical line parallel to the plate.

Taking an origin at any point of the vertical plate, an axis OY vertically upwards, and taking into account the assumptions just stated, we may write down the equation of motion of any particle, or layer of fluid, in the form

$$\frac{dv}{dt} = \nu \frac{d^2v}{dx^2} - g, \quad \dots \quad (1)$$

ν being the kinematic viscosity of the fluid.

Case I.—In the first case of motion to be considered the initial and boundary conditions to be fulfilled by v are: $v=0$, at $t=0$; $v=0$, at $x=0$; $v=-gt$, at $x=\infty$. If we put $v=v_1-gt$, we find that the conditions to be satisfied by v_1 are, then,—the usual equation representing diffusion of velocity,

$$\frac{dv_1}{dt} = \nu \frac{d^2v_1}{dx^2}, \quad \dots \quad (2)$$

together with

$$v_1=0, \text{ at } t=0; \quad v_1=gt, \text{ at } x=0; \quad v_1=0, \text{ at } x=\infty, \quad (2')$$

and the required solution, obtained from a well-known solution of a problem in heat conduction *, is

$$v_1 = \frac{gx}{2\sqrt{\pi\nu}} \int_0^t dq q e^{-z^2/(t-q)} \\ = -\frac{d}{dx} \left\{ g \left(\frac{\nu}{\pi} \right)^{\frac{1}{2}} \int_0^t dq q e^{-z^2/(t-q)} \right\}, \quad (3)$$

where $z^2 = x^2/\{4\nu(t-q)\}$. This yields the value required for v . One form in which the result may be written is

$$v = gt \left[2\xi^2 - \frac{2}{\sqrt{\pi}} \left\{ (1+2\xi^2) \int_0^\xi d\xi e^{-\xi^2} + \xi e^{-\xi^2} \right\} \right], \\ \text{where } \xi^2 = x^2/4\nu t, \quad (4)$$

which is suitable for numerical calculations when t is not excessively small. Since v represents the upward velocity of a particle of fluid at instant t , an expression representing the descent of any particle from rest at the

* Phil. Mag. iii. p. 787 Suppt., April 1927.

level of the origin is readily obtained by integration with respect to t . This curve of descent, which we may regard as the equation to the profile of the film, is given approximately by the equation

$$-y = -\frac{gx^2t}{2\pi^{\frac{1}{2}}\nu} \left\{ \pi^{\frac{1}{2}} - \frac{4}{3}\xi^{\frac{1}{2}} - \frac{4}{3}\xi^{\frac{3}{2}} - \frac{2}{15}\xi^{\frac{5}{2}} - \dots \right\}. \quad (5)$$

It appears from this that as t increases the profile is represented closely by the first three terms of (5) alone, the second term ($4gxt^{\frac{3}{2}}/3\pi^{\frac{1}{2}}\nu^{\frac{1}{2}}$) being ultimately the most important. The curve represented by (5) meets the plate at a small angle to the vertical drawn downwards, and its concavity is upwards.

Case II.—A quite different initiation of the fluid motion from that considered above is obtained by assuming as an initial condition that a film of uniform thickness, a , is given in contact with the vertical plate, and is allowed to descend under gravity starting from rest at instant $t=0$. At the surface of this film exposed to the atmosphere, $x=a$, since we may assume that viscous resistance due to the atmosphere is negligible, the boundary condition to be fulfilled is

$$\frac{dv}{dx} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

while v must still satisfy equation (1) above. For v_1 , defined as before, the conditions to be fulfilled are now

$$v_1 = 0, \text{ at } t=0; \quad v_1 = gt, \text{ at } x=0; \quad \frac{dv_1}{dx} = 0, \text{ at } x=a, \quad (6')$$

in addition to equation (2) above.

The determination of v_1 to fulfil these conditions can now be effected by the methods explained in the author's papers on heat-conduction problems*. Starting from the wave-train solution of (2),

$$v_1 = \theta e^{ikt - i\lambda x}, \text{ in which } \left(\frac{ik}{\nu}\right)^{\frac{1}{2}} = i\lambda, \quad \theta = \text{constant}, \quad (7)$$

we obtain first a periodic solution of (2) to satisfy the conditions

$$v_1 = \theta e^{ikt}, \text{ at } x=0; \quad \frac{dv_1}{dx} = 0, \text{ at } x=a. \quad . \quad . \quad (8)$$

* Phil. Mag. ix. (February 1930).

The wave-train (7) satisfies the first condition only. The two wave-trains

$$v_1 = \theta \{ e^{ikt - i\lambda x} + A e^{ikt - i\lambda(2a - x)} \} \quad . \quad . \quad . \quad (9)$$

taken together satisfy the second condition provided the coefficient A is given the value $A=1$, but leave unfulfilled the first condition. The latter condition is again fulfilled by the three wave-train solution

$$v_1 = \theta \{ e^{ikt - i\lambda x} + e^{ikt - i\lambda(2a - x)} - e^{ikt - i\lambda(2a + x)} \}, \quad . \quad (10)$$

which, however, in turn violates the condition imposed at end $x=a$. Each continuing wave-train is thus added in succession to satisfy an end condition. The sum of all terms so obtained is the required periodic solution. It may be written in the form

$$v_1 = \theta \left\{ \sum_0^{\infty} (-1)^n e^{ikt - i\lambda(2na + x)} - \sum_1^{\infty} (-1)^n e^{ikt - i\lambda(2na - x)} \right\}. \quad (11)$$

We next proceed to obtain from this an instantaneous solution of (2) to satisfy the conditions

$$v_1 = 0, \text{ at } t=0; \quad v_1 = \theta, \text{ at } x=0, \text{ at } t=0;$$

$$\frac{dv_1}{dx} = 0, \text{ at } x=a. \quad . \quad (12)$$

This instantaneous solution is $\frac{1}{\pi} \int_0^{\infty} dk v_1$ (as given by (11)),

that is,

$$\begin{aligned} v_1 &= \frac{\theta}{2\sqrt{\pi\nu t^3}} \left[\sum_0^{\infty} (-1)^n (2n+x) e^{-\frac{(2na+x)^2}{4\nu t}} \right. \\ &\quad \left. - \sum_1^{\infty} (-1)^n (2na-x) e^{-\frac{(2na-x)^2}{4\nu t}} \right] \\ &= \frac{\theta}{2\sqrt{\pi\nu t^3}} \sum_{-\infty}^{+\infty} (-1)^n (x+2na) e^{-\frac{(x+2na)^2}{4\nu t}} \quad . \quad . \quad . \quad (13) \end{aligned}$$

This result can be put into a more convenient form by means of a well-known theorem. Taking S , S_1 , S_2 to represent respectively the summation, sum of positive terms, sum of negative terms, on the right-hand side of (13), we obtain by the aid of the theorem

$$S_1 + S_2 = \frac{2\nu\pi\theta}{a^2} \sum_1^{\infty} n e^{-\lambda_n^2 \nu t} \sin \lambda_n x, \quad \lambda_n = \frac{n\pi}{a}, \quad . \quad (14)$$

If we put $S_1 + S_2 = f(a)$, then $S_1 = f(2a)$ and $S = S_1 - S_2$, so that, since $f(a)$ comprises simply the even terms of $2f(2a)$, we obtain finally

$$v_1 = \frac{\nu\pi\theta}{a^2} \sum_1^{\infty} n e^{-\lambda_n^2 \nu t} \sin \lambda_n x, \quad \lambda_n = \frac{n\pi}{2a}, \quad . \quad . \quad (15)$$

in which the summation refers to odd values of n only. To pass from (15) to the solution which fulfils the condition $v_1 = gt$ at the surface of the plate, we have merely to replace θ by $gq dq$, t by $(t - q)$ and integrate with respect to q from 0 to t . This process yields

$$v_1 = \frac{g\nu\pi}{a^2} \sum_1^{\infty} n \sin \lambda_n x \left[\frac{t}{\lambda_n^2 \nu} - \left(\frac{1}{\lambda_n^2 \nu} \right)^2 (1 - e^{-\lambda_n^2 \nu t}) \right], \quad (16)$$

and $v = v_1 - gt$, as before. Following the ordinary Fourier method, we may expand gt in a series of the type appearing above, namely,

$$gt = \sum_1^{\infty} A_n \sin \lambda_n x. \quad . \quad . \quad . \quad (17)$$

We find readily that $A_n = (4gt/n\pi)$, which means that gt is represented by the first summation in (16) and that

$$v = -\frac{16ga^2}{\pi^3 \nu} \sum_1^{\infty} \frac{1}{n^3} \sin \lambda_n x (1 - e^{-\lambda_n^2 \nu t}). \quad . \quad . \quad (18)$$

By applying the same method, we may verify that the velocity at each point within the film, when steady motion has ultimately been established, $(t \rightarrow \infty)$, is represented by

$$v = \frac{g}{\nu} \left(\frac{1}{2} x^2 - ax \right), \quad . \quad . \quad . \quad (19)$$

the solution upon which Jeffreys bases his treatment of the problem under consideration. Hence (18) is equivalent to

$$v = \frac{g}{\nu} \left(\frac{1}{2} x^2 - ax \right) + \frac{16ga^2}{\pi^3 \nu} \sum_1^{\infty} \frac{1}{n^3} \sin \lambda_n x e^{-\lambda_n^2 \nu t}. \quad . \quad (20)$$

The net distance through which any particle, or layer, of the fluid descends in time t from the start of motion is again found by an integration with respect to t . Taking $-y$ to represent the descent of the layer at distance x from the vertical plate, we find from (20) that

$$-y = \frac{g}{\nu} \left(ax - \frac{1}{2} x^2 \right) t - \frac{64ga^4}{\pi^5 \nu^2} \sum_1^{\infty} \frac{1}{n^5} \sin \lambda_n x (1 - e^{-\lambda_n^2 \nu t}); \quad (21)$$

and since when t is indefinitely large the exponential terms tend to disappear, we obtain as the equivalent of (21) for $t \rightarrow \infty$.

$$-y = \frac{g}{\nu} (ax - \frac{1}{2}x^2)t - \frac{g}{24\nu^2} (8a^3x - 4ax^3 + x^4). \quad (22)$$

The expressions (21) and (22) represent a surface within the film containing at each instant t the fluid particles which started from rest in a horizontal plane passing through the origin. It should be carefully noted, however, that throughout the motion contemplated in the above theory this surface is never free from viscous drag applied by the upper fluid in contact with it, and does not therefore correspond with an actual "free surface." The same remark applies to (5) above. The above treatment refers strictly to a film extending to infinity in the vertical direction and retaining a uniform thickness a as it descends. In the case of any finite film, initially of uniform thickness a , the condition of uniformity in thickness must be rapidly departed from near the top of the film, and it may be regarded as doubtful to what extent the curve of descent is to be relied upon as an indication of the shape of the actual profile. A complete investigation, taking into account the conditions to be fulfilled at a free surface, would be difficult, and it is probable that the results obtained above are as trustworthy as the fundamental assumptions themselves upon which the investigation depends.

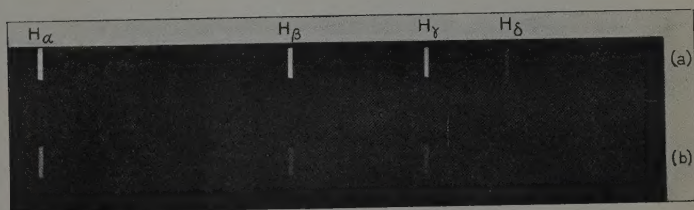
It is very desirable that an adequate test of the agreement obtainable between the above theoretical results and results of experiments should be made; but it could not be made with the data at the author's disposal. Certain experimental results have already been found to be in good agreement with Jeffreys' expression for the profile curve, which is derived from equation (19) alone. In (21) and (22) above there now appears an additional term. On the other hand, in the experimental paper referred to on page 730 the remark is made:—"To get agreement with theory it was found necessary to add one fringe thickness of liquid to the film at all points." The additional term in (22) may satisfactorily account for the imperfect agreement here referred to and "for the non-parabolic shape of the profile found experimentally."

The agreement with theory which has so far been obtained obviously implies agreement with the theoretical results obtained for the case of the film of uniform thickness, initially at rest, allowed to descend under gravity—the second problem treated above. In some of the actual experiments described, however, the conditions under which the film is produced correspond closely with the initial conditions assumed in the first problem treated above, and it is interesting to notice that certain experiments in which the plate was “kept immersed up to the time of withdrawal did not give as good agreement” with equation (19). The above results illustrate the importance of the initial conditions of experiment in determining the ultimate shape of the profile obtained. Those experiments in which the main body of the liquid is rapidly lowered from the vertical plate to form the film, and which nevertheless give approximate agreement with theory, seem to indicate that at an early stage in the experiment a fairly uniform film is produced. Yet according to the continuous fluid motion theory of the present paper it would be impossible to account for a transition from the motion determined in the first to that determined in the second of the problems discussed above. The conditions determining the appropriate thickness (a) in each case are still obscure. Apart from a reference to a “concavity upwards” appearing in the earlier part of the profile curves, no definite evidence of a profile of the type indicated in equation (5) has apparently been obtained. From the above theoretical result it is certainly to be expected, and further experimental investigations may confirm this.

The writer desires to state that his attention was drawn to this subject by Professor T. Alty of this University, who kindly obtained for him all the required references. He has much pleasure also in acknowledging the interest taken by Prof. Alty in the investigations recorded and in the preparation of the paper.

Some interesting cognate problems, relating to the motion of a viscous fluid, either due to gravity or due to the motion of a plate in contact with the fluid, for cases in which the plate is finite, can be readily solved by the method illustrated in the present paper. These are, meantime, reserved for another opportunity.

[The Editors do not hold themselves responsible for the views expressed by their correspondents.]



Spectrograms of Balmer Series.

- (a) Exposure 40 mins.
(b) Exposure 2 mins.

FIGS. 1-6.—Electron diffraction photographs of oxide film from aluminium.



FIG. 1.—As prepared.



FIG. 2.—Same specimen heated at 690° C. for 10 hours.



FIG. 3.—Same specimen heated at 680° C. for a further 10 hours.



FIG. 4.—Same specimen heated at 680° C. for a further 50 hours.

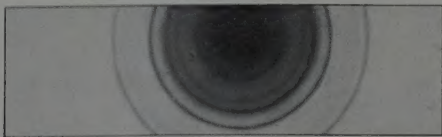


FIG. 5.—Same specimen heated at 800° C. for a further 2 hours

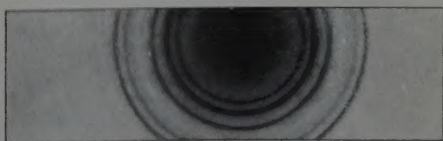


FIG. 6.—Same specimen heated at 1000° C. for a further 1 hour

